**DEFLATED GAINS AND NUMERAIRE INVARIANCE: A SIMPLE EXPOSITION**

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## 1. Exchange rates, numeraires, and deflators

**Exchange rate example.** If you buy a security at the beginning of the period and sell it at the end of the period, it is easy to compute the increase or decrease in value. But suppose you have a friend from abroad who buys and sells the same security. What is the change in the value of your friend's investment measured in the foreign currency? You will need to use the exchange rate to compute this value.

Let $V$ be the price of the security denominated in dollars. At the beginning of the period, the value of the security is $V_0$, and the end of the period it is $V_1$. The change in the value is $dV = V_1 - V_0$. To keep track of dividends, we will use the following accounting device. Let $D$ be the *cumulative dividends* denominated in dollars, with values $D_0$ and $D_1$ at the beginning and end of the period. The dividends that accrue to the owner of the security during the period are given by $dD = D_1 - D_0$. The *gain* is defined as $G = V + D$, with $G_0$ and $G_1$ as values at the beginning and end of the period. Therefore the change in the dollar value of an investment in the security is given by the change in the gain, which is composed of capital gains and dividends:

$$dG = dV + dD = (V_1 - V_0) + (D_1 - D_0).$$

As noted above, in order to compute the change in the foreign-currency value of the investment, we need the exchange rate. Let $Y$ be the exchange rate with $Y_0$ and $Y_1$ beginning and ending values. The foreign-currency value of the security is $V^Y = Y\, V$: The cost of the security in foreign currency (at the beginning of the period) is $V_0^Y = Y_0\, V_0$, and the end-of-period foreign-currency value of the security is $V_1^Y = Y_1\, V_1$. What about the dividends? The end-of-period foreign-currency value of the dividend flow is $Y_1\,(D_1 - D_0) = Y_1\, dD$. The change in the foreign-currency value of the investment is

$$(Y_1\, V_1 - Y_0\, V_0) + Y_1\, dD. \tag{1}$$

The first term is the change in the value of the security and the second term is the value of the change in the cumulative dividends.

Now we want to be able to write the change in the foreign-currency value of the investment as given in (1) in parallel fashion to the way we wrote it for the dollar

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value. In particular, let $G^Y$ be the gain denominated in foreign-currency. We want to define $D^Y$ in such a way that $G^Y \equiv V^Y + D^Y$ and

$$dG^Y \equiv (Y_1\,V_1 - Y_0\,V_0) + Y_1\,dD$$

hold identically. The possibility that $V_0 = V_1 = 0$ implies

$$dD^Y = Y_1\,dD,$$

where $D_0^Y$ is arbitrary. The value of $D_1^Y$ is given by $D_1^Y = D_0^Y + dD^Y$. It is convenient to let $D_0^Y = Y_0\,D_0$. We can write the changes in the deflated values as follows:

$$\begin{aligned} dV^Y &= Y_0\,dV + V_0\,dY + dV\,dY \\ dD^Y &= Y_0\,dD + dD\,dY. \end{aligned}$$

**Deflators and numeraires.** A *deflator* is a strictly positive process. We can renormalize security prices by multiplying them by a deflator. The exchange rate is an example of a deflator. (Below we will see other examples.) Thus we refer to $V^Y$ as the deflated value of the security and $G^Y$ as the *deflated gain*.

## 2. Trading strategies and arbitrage

A *trading strategy* is a set of portfolio weights. A *self-financing* trading strategy is a trading strategy that has net cash-flows at only two points in time, now and later. In the simple setup in this exposition, a self-financing trading strategy is simply the purchase or sale of the single asset at the beginning of the period and the subsequent unwinding of the transaction at the end of the period.

Let $\theta$ be the number of shares of the security bought, with $\theta < 0$ if the security is sold short. The position at the beginning of the period is $\theta_0\,V_0$ and at the end of the period is $\theta_1\,V_1$. In general the relation between the two is

$$\theta_1\,V_1 = \theta_0\,V_0 + \big(\theta_0\,dV + V_0\,d\theta + dV\,d\theta\big). \tag{2}$$

With a self-financing trading strategy we have

$$\theta_1\,V_1 = \theta_0\,V_0 + \theta_0\,dG. \tag{3}$$

Together (2) and (3) imply

$$\theta_0\,dD = V_1\,d\theta. \tag{4}$$

The left-hand side of (4) shows the dividends that must be reinvested, while the right-hand side shows how the weight must change to effect the reinvestment. In particular, if no dividends are paid, no change in the weight is required.

An *arbitrage* is a self-financing trading strategy that produces something for nothing. Arbitrages involve changes in sign. The following are arbitrages: $(i)$ $\theta_0\,V_0 \leq 0$ and $\theta_1\,V_1 \gg 0$ or $(ii)$ $\theta_0\,V_0 < 0$ and $\theta_1\,V_1 \gg 0$, where $a \gg b$ denotes $a > b$ over all possible outcomes. In either case, $\theta_0\,dG \gg 0$.

*Numeriare invariance* is the proposition that changing the units in which gains are measured has no impact on whether a trading strategy is self-financing or not. Multiplying both sides of (4) by $Y_1$ produces

$$\theta_0 \, dD^Y = V_1^Y \, d\theta,$$

confirming the proposition. An implication of numeraire invariance is that changing the units in which gains are measured has no impact on the existence of arbitrages. Let us return for a moment to the foreign-exchange example. Suppose that our investment had zero cost in dollars at the beginning of the period and was sure to have a positive value in dollars at the end of the period. This would be an arbitrage: We could get something for nothing. But what if we measure things in terms of the foreign currency? Will there still be an arbitrage? Of course there would.

**The absence of arbitrage.** Numeraire invariance leads to the following central proposition. If there is a numeraire such that the deflated gain is a martingale (for all self-financing trading strategies), then there are no arbitrage opportunities. Let $E_0[\,\cdot\,]$ denote the conditional expectation operator. If the gain is a martingale, then $E_0[dG^Y] = 0$, which makes it impossible to guarantee a change in sign, thereby ruling out all arbitrages. If there is such a deflator, it is known as the *state–price deflator*. Since the state–price deflator plays a special role, we give it a special symbol, $\pi$, and we write the absence-of-arbitrage condition as $E_0[dG^\pi] = 0$.

A bit of algebra allows us to rewrite $E_0[dG^\pi] = 0$ as

$$E_0[dV] + E_0[dD] = V_0 \, E_0\left[\frac{-d\pi}{\pi_0}\right] + E_0\left[dV\left(\frac{-d\pi}{\pi_0}\right)\right] + E_0\left[dD\left(\frac{-d\pi}{\pi_0}\right)\right]. \tag{5}$$

The left-hand side of (5) is the expected change in the value of the investment. The right-hand side involves the risk-free interest rate and risk premia.

*A simple example.* Here is a simple example that illustrates one feature of the foregoing. Consider an asset that pays a dividend at the end of the period. Let the ex-dividend value of the asset be identically zero: $V_1 \equiv 0$. In this case $dV \equiv -V_0$, and (5) can be written as follows:

$$\begin{aligned} V_0 &= \frac{1}{\pi_0} \, E_0[\pi_1 \, dD] \\ &= \frac{1}{\pi_0} \big(E_0[\pi_1] \, E_0[dD] + C_0[d\pi, dD]\big), \end{aligned} \tag{6}$$

where $C_0[\,\cdot\,,\,\cdot\,]$ is the conditional covariance operator. Equation (6) shows that there is a risk premium associated with the uncertainty of the cummulative dividends unless cummulative dividends are locally riskless.

*A binomial example.* Let

$$\begin{aligned}
\frac{dV}{V_0} &= \mu_V\,\Delta + \sigma_V\,\varepsilon\,\sqrt{\Delta} \\
\frac{dD}{V_0} &= \mu_D\,\Delta + \sigma_D\,\varepsilon\,\sqrt{\Delta} \\
\frac{d\pi}{\pi_0} &= \mu_\pi\,\Delta + \sigma_\pi\,\varepsilon\,\sqrt{\Delta},
\end{aligned}$$

where $\Delta$ is the length of the period and where

$$\varepsilon = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2}. \end{cases}$$

Note that

$$E_0\left[\frac{dV}{V_0}\right] = \mu_V\,\Delta, \quad E_0\left[\frac{dD}{V_0}\right] = \mu_D\,\Delta, \quad \text{and} \quad E_0\left[\frac{-d\pi}{\pi_0}\right] = -\mu_\pi\,\Delta,$$

and

$$\begin{aligned}
E_0\left[\frac{dV}{V_0}\left(\frac{-d\pi}{\pi_0}\right)\right] &= -\sigma_V\,\sigma_\pi\,\Delta - \mu_V\,\mu_\pi\,\Delta^2 \\
E_0\left[\frac{dD}{V_0}\left(\frac{-d\pi}{\pi_0}\right)\right] &= -\sigma_D\,\sigma_\pi\,\Delta - \mu_D\,\mu_\pi\,\Delta^2.
\end{aligned}$$

Thus we can write (5) as

$$\mu_V + \mu_D = -\mu_\pi - \sigma_V\,\sigma_\pi - \sigma_D\,\sigma_\pi - (\mu_V + \mu_D)\,\mu_\pi\,\Delta,$$

where the last term on the right-hand side goes to zero as the length of the period goes to zero.

## 3. Lognormally distributed asset prices

In this section, we adopt a specfic class of discrete-time processes. Let

$$dV = V_0\left(\exp\left\{\widetilde{\mu}_V\,\Delta + \sigma_V\cdot\varepsilon\,\sqrt{\Delta}\right\} - 1\right) \tag{7a}$$

$$dD = V_0\left(\exp\left\{\widetilde{\mu}_D\,\Delta + \sigma_D\cdot\varepsilon\,\sqrt{\Delta}\right\} - 1\right) \tag{7b}$$

$$d\pi = \pi_0\left(\exp\left\{\widetilde{\mu}_\pi\,\Delta + \sigma_\pi\cdot\varepsilon\,\sqrt{\Delta}\right\} - 1\right), \tag{7c}$$

where $\Delta$ is the length of the period and $\varepsilon$ is a vector of independent normally distribtuted shocks. Note that

$$d\log(V) = \widetilde{\mu}_V\,\Delta + \sigma_V\cdot\varepsilon\,\sqrt{\Delta},$$

for example. In order to evaluate the expectations, we will use the following fact: If $x \sim \mathcal{N}(\mu,\sigma)$, then $E[e^x] = \exp\left\{\mu + \frac{1}{2}\,\|\sigma\|^2\right\}$. Also we use the following first-order approximation: $e^{x\,\Delta} - 1 = x\,\Delta + \mathcal{O}(\Delta^2)$. Assuming $V_0 > 0$, we can write

$$E_0\left[\frac{dV}{V_0}\right] \doteq \left(\widetilde{\mu}_V + \frac{1}{2}\,\|\sigma_V\|^2\right)\Delta \qquad \text{and} \qquad E_0\left[\frac{dV}{V_0}\left(\frac{-d\pi}{\pi_0}\right)\right] \doteq -\sigma_V\cdot\sigma_\pi\,\Delta,$$

for example. Define

$$\mu_i = \widetilde{\mu}_i + \frac{1}{2}\,\|\sigma_i\|^2 \qquad \text{for } i = V, D, \pi.$$

Then we can write (5)

$$\mu_V + \mu_D = -\mu_\pi - \sigma_V \cdot \sigma_\pi - \sigma_D \cdot \sigma_\pi, \tag{8}$$

where we have canceled $\Delta$ from both sides.

To see how the risk-free rate is involved, we can use (8) to examine a one-period, risk-free zero-coupon bond. It is convenient to model the payoffs to this bond as follows: The bond's dividends are identically zero, and its end-of-period value is identically one. In terms of the parameters, we have $\widetilde{\mu}_D = 0$, $\sigma_D = \sigma_V = 0$, and $\widetilde{\mu}_V = r$, the risk-free rate. In this case we have $\mu_V = -\mu_\pi$. Since the return on the bond defines the one-period risk-free interest rate, we have shown that $E_0[-d\pi/\pi_0]$ equals the risk-free rate. The last two terms on the right-hand side of (8) are the risk premia that stem from the covariance between changes in the state–price deflator and changes in the securities price and/or its dividends. Note, however, that dividends are typically modeled in such a way that $\sigma_D \cdot \sigma_\pi = 0$.

**Foreign-currency state–price deflator.** Since the expected relative change in the state–price deflator is the risk-free interest rate for investments in dollars, there must be a different state–price deflator for investments measured in foreign currency. Letting $Y$ be the exchange rate, the foreign-currency state–price deflator is given by $\varpi = \pi/Y$.

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