RISK-NEUTRAL PRICING AND RISK NEUTRALITY: A TALE OF TWO EQUIVALENT MARTINGALE MEASURES

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Abstract. Two properties of asset prices have often been associated with risk-neutrality: (a) the expected return on all assets equals the risk-free rate (the *expected-return property*) and (b) the value of an asset equals the present value of the expectation (under the physical measure) of its payoff (the *certainty-equivalent property*). In this paper, we show the following: (i) the two properties are equivalent when interest rates are deterministic but mutually exclusive when interest rates are stochastic, (ii) an economy of risk-neutral investors who consume only at a single point of time in the future will support the certainty-equivalent property, (iii) the distribution that can be uncovered using options prices is associated with the certainty-equivalent property, (iv) the distribution associated with the expected-return property cannot be uncovered using option prices when interest rates are stochastic, although they can be uncovered from futures prices on options that expire on the delivery date. We provide an empirical investigation of the relationship between these measures in the context of multi-factor models of the term structure.

1. Introduction

The idea of risk-neutral pricing dates back to the analysis of the option prices by Cox and Ross (1976). They address a class of models in which interest rates are deterministic. They propose what amounts to the Feynman–Kac solution technique to the partial differential equation (PDE) derived by Black and Scholes (1973). Cox and Ross (1976) note that since this PDE does not involve any risk-preference parameters, *any* assumption about risk preferences that leads to a solution is valid. As they say (p. 153):

> A convenient choice of preferences ... is risk neutrality. In such a world equilibrium requires that the expected returns on [assets] must equal the riskless rate.

Date: October 8, 1996.
JEL Classification. G12.
Key words and phrases. Equivalent martingale measures, risk-neutral pricing, stochastic interest rates, Black–Scholes, option pricing, asset pricing.
The views expressed herein are the authors' and do not necessarily reflect those of the Board of Governors or the Federal Reserve System.

This property of asset prices has been widely associated with risk-neutrality in the asset pricing literature.[1] We refer to this property as the *expected-return property.* Because interest rates are deterministic in their setting, Cox and Ross (1976) were able to derive a representation for asset prices in terms of the present value of a certainty equivalent. This property, too, has been widely associated with risk-neutrality. We refer to this property as the *certainty-equivalent property.*

However, as Cox, Ingersoll, Jr., and Ross (1981, CIR) point out, these properties do not necessarily follow from risk neutrality when when interest rates are stochastic. They make the following observation regarding the expected-return property:

> Since there are no term premiums in this formulation, many authors have been lured into referring to it as the "Risk-Neutral Expectations Hypothesis." In fact, [it] is not a consequence of universal risk-neutrality . . . .

Indeed, CIR show that risk-neutrality itself—as commonly modeled—produces deterministic interest rates. In addition, CIR show that in an economy with both risk-neutral agents and stochastic interest rates, there is a risk premium built into expected returns. In particular, they derive the expected rate of return for zero-coupon bonds when risk-neutral investors consume only at a single point of time in the future. They also describe conditions under which the expected-return property holds absent risk neutrality.

In the meantime, the basic notions of risk-neutral pricing were extended to more general settings by Harrison and Kreps (1979). Modern asset pricing theory can be characterized as *equivalent martingale pricing*, since deflated asset prices are martingales under an equivalent measure. Moreover, it turns out that options prices can be used to uncover the distribution of the underlier one such equivalent martingale measure. It has been asserted that if the representative individual were risk neutral, then we could identify this distribution with the physical measure and use it, for example, to calculate market expectations.

To establish the link between asset prices and risk preferences, the following characterization of the absence of arbitrage is useful: There exists a state-price deflator with the property that deflated asset prices are martingales under the physical measure.[2] Thus the properties of asset prices are related to the properties of the state-price deflator. Under fairly general conditions, the state-price deflator can be interpreted as the marginal utility of wealth (or, absent corner solutions, the marginal utility of consumption).

In this paper, we show the following: (i) the two properties are equivalent when interest rates are deterministic and they are mutually exclusive when interest rates are stochastic, (ii) the link between the certainty-equivalent property and risk neutrality is robust, but the agent's preferences are quite unappealing (risk-neutral

[1] Except in a one-good, one-date economy, risk neutrality is an ambiguous notion. See various cites [to be included].

[2] Duffie (1996, chapter 6).

investors consume only at a single point of time in the future), (*iii*) the distribution that can be uncovered using options prices is associated with the certainty-equivalent property, (*iv*) the distribution associated with the expected-return property cannot be uncovered using option prices when interest rates are stochastic, although they can be uncovered from futures prices on options that expire on the delivery date.

The analysis of this paper amounts to an application of equivalent martingale measure pricing. The rest paper is organized as follows. First, we take the existence of a state-price deflator as our starting point and derive the absence-of-arbitrage condition for the expected return under the physical measure, and we show that the expected return depends on the numeraire. Second, we show by normalizing asset prices with the value of a given asset, one can generate an equivalent measure associated with that deflator–asset under which deflated asset prices are martingales. Thus different deflator–assets generate different measures with different representations for asset prices and different dynamics for asset prices under its associated measure. Third, we characterize two important equivalent martingale measures—the futures measure (associated with the money-market account) and the forward measure (associated with a zero-coupon bond). The two are identical when interest rates are deterministic. Fourth, we show the relationship between the futures measure and the Feynman–Kac solution technique, and we show how the futures measure can be uncovered with futures prices and the forward measure can be uncovered with forward prices (or options prices). Fifth, we examine risk-neutral pricing in terms of the futures and forward measures when interest rates are random. Each of the two measures has a claim to the title: Under the futures measure the expected return on assets equals the risk-free rate, while under the forward measure asset prices equal the present value of a certainty equivalent. Finally, we illustrate these points with a multi-factor model of the term structure.

2. The state-price deflator

Absence of arbitrage. Let $V(t)$ be a security price process and $Y(t)$ be a deflator. A *deflator* is a strictly positive process. Then a deflated security price process is given by $V(t)\,Y(t)$. The absence of arbitrage is equivalent to the existence of a *state-price deflator* $m(t)$ where, for all $s \geq t$, deflated asset prices are martingales under the physical measure:

$$E_t^{\mathcal{P}}[m(s)\,V(s)] = m(t)\,V(t). \tag{2.1}$$

We note in passing that if markets are complete, the state-price deflator is unique given the numeraire.

Since $m(t)$ is strictly positive, we can write its process as

$$\frac{dm(t)}{m(t)} = \mu_m(t)\,dt + \sigma_m(t)^\top dW(t), \tag{2.2}$$

where $W(t)$ is a vector of orthogonal Brownian motions under the physical measure $\mathcal{P}$. We suppose that $V(t)$ is the value of an asset that pays no dividends.[3] Then we can write

$$dV(t) = \bar{\mu}_V(t)\, dt + \bar{\sigma}_V(t)^\top dW(t), \tag{2.3}$$

where the bars over $\bar{\mu}_V(t)$ and $\bar{\sigma}_V(t)$ indicate that these are absolute rather than relative drifts and volatilities. We assume there is a *money-market account* with value process $d\beta(t) = r(t)\,\beta(t)\,dt$, where $r(t)$ is the instantaneous risk-free rate. Note that we can write $\beta(t) = \exp\left(\int_0^t r(u)\,du\right)$, given $\beta(0) = 1$.

Given the dynamics of $m(t)$ and $V(t)$, Ito's lemma gives the dynamics for $m(t)\,V(t)$ as

$$d\big(m(t)\,V(t)\big) = m(t)\left\{\mu_V(t) + \mu_m(t)\,V(t) + \sigma_m(t)^\top \bar{\sigma}_V(t)\right\} dt + m(t)\left\{\sigma_m(t) + \bar{\sigma}_V(t)\right\}^\top dW(t). \tag{2.4}$$

The absence of arbitrage as characterized by (2.1) implies that the drift of $m(t)\,V(t)$ in (2.4) is zero. Thus we have

$$\bar{\mu}_V(t) = -\mu_m(t)\,V(t) - \sigma_m(t)^\top \bar{\sigma}_V(t). \tag{2.5}$$

Applying (2.5) to the money-market account produces $\mu_m(t) = -r(t)$. In addition, it is conventional to define the *market price of risk* as $\lambda(t) := -\sigma_m(t)$. With this change of notation, we can write (2.2) as $dm(t)/m(t) = -r(t)\,dt - \lambda(t)^\top dW(t)$. Moreover we can write (2.5) as

$$\bar{\mu}_V(t) = r(t)\,V(t) + \lambda(t)^\top \bar{\sigma}_V(t), \tag{2.6}$$

Note that if $V(t)$ is strictly positive, we can write (2.6) as

$$\mu_V(t) = r(t) + \lambda(t)^\top \sigma_V(t), \tag{2.7}$$

where $\mu_V(t) := \bar{\mu}_V(t)/V(t)$ and $\sigma_V(t) := \bar{\sigma}_V(t)/V(t)$. Equation (2.7) states that the expected return equals the risk-free rate plus a risk-premium that depends on the negative of the conditional covariance of the asset's return with the state-price deflator.

Changing the numeraire. Suppose we change the numeraire by deflating by another deflator, $Y(t)$. Let asset values denominated in the new units be denoted by $\widetilde{V}(t) := Y(t)\,V(t)$. Note that (2.1) implies

$$E_t^{\mathcal{P}}[\widetilde{m}(s)\,\widetilde{V}(s)] = \widetilde{m}(t)\,\widetilde{V}(t), \tag{2.8}$$

where $\widetilde{m}(t) := m(t)/Y(t)$ is the state-price deflator for the new numeraire. Therefore we can write the absence-of-arbitrage condition in terms of the new numeraire as $\bar{\mu}_{\widetilde{V}}(t) = \widetilde{r}(t)\,\widetilde{V}(t) + \widetilde{\lambda}(t)^\top \bar{\sigma}_{\widetilde{V}}(t)$, and the dynamics of the new state-price deflator can be expressed in terms of the short rate and price of risk denominated in terms of the new numeraire: $d\widetilde{m}(t)/\widetilde{m}(t) = -\widetilde{r}(t)\,dt - \widetilde{\lambda}(t)^\top dW(t)$.

[3] $V(t)$ can be thought of as the value of a self-financing trading strategy under which any dividends paid are reinvested.

It is natural to use this set up when $m(t)$ is the state-price deflator denominated in domestic currency, $\widetilde{m}(t)$ is the state-price deflator denominated in foreign currency, in which case $Y(t)$ is the foreign exchange rate. Another natural use is when $m(t)$ is the state-price deflator denominated in real terms, while $\widetilde{m}(t)$ is the state-price deflator denominated in nominal terms, in which case $Y(t)$ is the price level.

From a modeling perspective, we are free to choose any two of $\{m(t), \widetilde{m}(t), Y(t)\}$ as exogenous and derive the dynamics of the third using Ito's lemma. For some purposes it is convenient to treat the processes for $m(t)$ and $\widetilde{m}(t)$ as exogenous and derive the process for $y(t) := \log(Y(t)) = \log(m(t)/\widetilde{m}(t))$:

$$dy(t) = \mu_y(t)\,dt + \sigma_y(t)^\top dW(t),$$

where

$$\mu_y(t) = \widetilde{r}(t) - r(t) - \frac{1}{2}\left(\|\widetilde{\lambda}(t)\|^2 - \|\lambda(t)\|^2\right) \tag{2.9a}$$

and

$$\sigma_y(t) = \widetilde{\lambda}(t) - \lambda(t). \tag{2.9b}$$

See, for example, Saá-Requejo (1994). Note that we can rewrite (2.9) as

$$\widetilde{r}(t) = r(t) + \mu_y(t) + \frac{1}{2}\|\sigma_y(t)\|^2 + \lambda(t)^\top \sigma_y(t). \tag{2.10}$$

Equation (2.10) can be interpreted as either (i) the relationship between nominal $(\widetilde{r})$ and real (r) interest rates, which involves expected inflation (μ_y), the variance of inflation $(\|\sigma_y\|^2)$ and a risk premium $(\lambda^\top\sigma_y)$, or (ii) the relationship between foreign $(\widetilde{r})$ and domestic (r) interest rates, which involves expected exchange-rate depreciation, and so forth.

3. Equivalent martingale measures and deflator–assets

Deflator–assets. Let $z(t)$ be the value of a strictly positive asset that pays no dividends over the relevant period (a self-financing trading strategy). Since $z(t)$ is strictly positive, we can write its process as $dz(t)/z(t) = \mu_z(t)\,dt + \sigma_z(t)^\top dW(t)$. Define $\xi_z(t) := m(t)\,z(t)$. Since $z(t)$ is the value of an asset, we have by (2.7)

$$\frac{d\xi_z(t)}{\xi_z(t)} = -\theta_z(t)^\top dW(t), \tag{3.1}$$

where $\theta_z(t) := \lambda(t) - \sigma_z(t)$. Given the absence of arbitrage and (3.1), we can define an equivalent measure $\mathcal{Q}_z$ using Girsanov's Theorem as follows:

$$dW_z(t) = dW(t) + \theta_z(t)\,dt, \tag{3.2}$$

where $W_z(t)$ is a vector of standard, independent Brownian motions under $\mathcal{Q}_z$. The dynamics under $\mathcal{Q}_z$ of an arbitrary variable $X(t)$ can be easily found given its dynamics under $\mathcal{P}$ using (3.2):

$$\begin{aligned} dX(t) &= \bar{\mu}_x(t)\,dt + \bar{\sigma}_X(t)^\top dW(t) \\ &= \bar{\mu}_x(t)\,dt + \bar{\sigma}_X(t)^\top \big(dW_z(t) - \theta_z(t)\,dt\big) \\ &= \big(\bar{\mu}_X(t) - \bar{\sigma}_X(t)^\top \theta_z(t)\big)\,dt + \bar{\sigma}_X(t)^\top dW_z(t). \end{aligned} \tag{3.3}$$

We now show that $\mathcal{Q}_z$ is an equivalent *martingale* measure. Let $Y(t) = z(t)^{-1}$ be a deflator and consider the deflated asset price $V_z(t) = V(t)\,Y(t) = V(t)/z(t)$. We refer to z as the *deflator–asset*. Ito's lemma gives the process for $V_z(t)$ under the physical measure $\mathcal{P}$:

$$dV_z(t) = \theta_z(t)^\top \bar{\sigma}_{V_z}(t)\,dt + \bar{\sigma}_{V_z}(t)^\top dW(t), \tag{3.4}$$

where $\bar{\sigma}_{V_z}(t) = \bar{\sigma}_V(t) - \sigma_z(t)$. Using (3.3) and (3.4) we can write

$$dV_z(t) = \bar{\sigma}_{V_z}(t)^\top dW_z(t),$$

which shows that asset values deflated by $Y(t) = z(t)^{-1}$ are martingales under the equivalent measure $\mathcal{Q}_z$.[4] Thus, for $s \geq t$, we have

$$V_z(t) = E_t^{\mathcal{Q}_z}[V_z(s)]. \tag{3.5}$$

Also note that $\xi_z(t)$ is the (conditional expectation of the) Radon-Nikodym derivative: $\xi_z(t) = E_t^{\mathcal{P}}[d\mathcal{Q}_z/d\mathcal{P}]$.[5] In particular, we have

$$E_t^{\mathcal{Q}_x}[X(s)] = \frac{E_t^{\mathcal{P}}[\xi_z(s)\,X(s)]}{\xi_z(t)} \tag{3.6}$$

for any time-s measurable random variable $X(s)$. Applying the same reasoning as above, it can be shown that

$$E_t^{\mathcal{Q}_\zeta}[X(s)] = \frac{E_t^{\mathcal{Q}_z}\left[\big(\zeta(s)/z(s)\big)\,X(s)\right]}{\big(\zeta(t)/z(t)\big)}, \tag{3.7}$$

where $\mathcal{Q}_\zeta$ is an equivalent martingale measure associated with the deflator–asset ζ.

Representation and dynamics of asset values. Given the equivalent martingale measure associated with a given deflator–asset, we can rewrite (3.5) to get a representation of asset prices under $\mathcal{Q}_z$:

$$V(t) = E_t^{\mathcal{Q}_z}\left[\frac{z(t)}{z(s)}\,V(s)\right]. \tag{3.8}$$

Note that (3.8) is valid for $z(t) = 1/m(t)$, in which case $\mathcal{Q}_z = \mathcal{P}$. Let $P(t,s)$ denote the price at time t of a default-free zero-coupon bond that pays one unit at time s. Then the term structure of interest rates has the following representation:

$$P(t,s) = E_t^{\mathcal{Q}_z}\left[\frac{z(t)}{z(s)}\right]. \tag{3.9}$$

[4]We have just shown that it is sufficient for the deflator to be the inverse of a strictly positive asset value. We show necessity in the Appendix.

[5]See Duffie (1996, chapter 6 and appendix D).

Although all equivalent martingale measures give the same value for an asset (as they must by construction), they do not imply the same dynamics for asset prices under their respective measures. In particular, using (2.3), (2.6), and (3.2), we have

$$dV(t) = \big(r(t)\,V(t) + \sigma_z(t)^\top \bar\sigma_V(t)\big)\,dt + \bar\sigma_V(t)^\top dW_z(t). \tag{3.10}$$

Again, if $V(t)$ is strictly positive, we can write the expected return under $\mathcal{Q}_z$ as

$$r(t) + \sigma_z(t)^\top \sigma_V(t). \tag{3.11}$$

Equation (3.11) shows that the expected return on an asset under an equivalent martingale measure equals the risk-free rate plus a premium equal to the conditional covariance of the asset's return with that of the deflator–asset. Thus, only if the deflator–asset is predictable—*i.e.* only if $\sigma_z(t) = 0$—will expected returns under $\mathcal{Q}_z$ equal the risk-free rate.

4. The futures measure and the forward measure

Two equivalent martingale measures that are particularly useful are the futures measure and the forward measure. These measures are associated with deflators derived from the money-market account and from a zero-coupon bond that matures on a given date, respectively.

The *futures measure* $\mathcal{Q}_\beta$ is associated with $z(t) = \beta(t)$. The representation for asset values under $\mathcal{Q}_\beta$ is

$$V(t) = E_t^{\mathcal{Q}_\beta}\left[\frac{\beta(t)}{\beta(s)}\,V(s)\right] = E_t^{\mathcal{Q}_\beta}\left[\exp\left(-\int_t^s r(u)\,du\right)V(s)\right], \tag{4.1}$$

and the term structure is given by

$$P(t,s) = E_t^{\mathcal{Q}_\beta}\left[\exp\left(-\int_t^s r(u)\,du\right)\right]. \tag{4.2}$$

Note that under $\mathcal{Q}_\beta$, expected returns equal $r(t)$ for all assets since $\sigma_z(t) = 0$ when the money-market account is the deflator asset. The futures measure thus embodies the expected-return property.

The *forward measure* $\mathcal{Q}_\tau$ is associated with

$$z(t) = \begin{cases} P(t,\tau) & \text{if } s \le \tau \\ \exp\left(\int_\tau^s r(u)\,du\right) & \text{if } s > \tau. \end{cases}$$

Note that under $\mathcal{Q}_\tau$ we have, for $s \le \tau$,

$$V(t) = E_t^{\mathcal{Q}_\tau}\left[\frac{P(t,\tau)}{P(s,\tau)}\,V(s)\right] = P(t,\tau)\,E_t^{\mathcal{Q}_\tau}\left[\frac{1}{P(s,\tau)}\,V(s)\right]. \tag{4.3}$$

For payoffs at time τ we have a particularly simple expression:

$$V(t) = P(t,\tau)\,E_t^{\mathcal{Q}_\tau}\left[V(\tau)\right]. \tag{4.4}$$

The forward measure thus embodies the certainty-equivalent property.

If we apply (4.4) to the money-market account and rearrange, we obtain

$$\frac{1}{P(t,\tau)} = E_t^{\mathcal{Q}_\tau}\left[\frac{\beta(\tau)}{\beta(t)}\right] = E_t^{\mathcal{Q}_\tau}\left[\exp\left(\int_{s=t}^\tau r(s)\,ds\right)\right]. \tag{4.5}$$

The term structure does not have a particularly useful representation under $\mathcal{Q}_\tau$:

$$P(t,s) = P(t,\tau)\, E_t^{\mathcal{Q}_\tau}\left[\frac{1}{P(s,\tau)}\right],$$

for $s \le \tau$. Note that expected returns under $\mathcal{Q}_\tau$ are *not* equalized across assets; rather there is a risk premium that depends on $\sigma_V(t)^\top \sigma_P(t,\tau)$, the conditional covariance of an asset's return with the return on the bond that matures at time τ.

Note that the two measures are different unless $\sigma_P(t,\tau) = 0$, in which case interest rates are deterministic and

$$\frac{\beta(t)}{\beta(s)} = \frac{P(t,\tau)}{P(s,\tau)} = \exp\left(-\int_t^s r(u)\,du\right),$$

so that the two measures are identical. In other words, if interest rates are deterministic, the two properties are equivalent, while if interest rates are stochastic, the two properties are inconsistent.

Forward prices and futures prices. Let the *forward price* at time t for delivery of $Y(\tau)$ at time τ be denoted $F(t,\tau)$, where $Y(\tau)$ is any time-τ measurable random variable. The forward price is what makes the value of the forward contract zero. Thus, for the futures measure, solving

$$E_t^{\mathcal{Q}_\beta}\left[\frac{\beta(t)}{\beta(\tau)}\big(Y(\tau) - F(t,\tau)\big)\right] = 0, \tag{4.6}$$

for $F(t,\tau)$ yields

$$F(t,\tau) = \frac{E_t^{\mathcal{Q}_\beta}\left[\exp\left(-\int_t^\tau r(u)\,du\right) Y(\tau)\right]}{P(t,\tau)}. \tag{4.7}$$

If $Y(t) = V(t)$ is the value of an asset that pays no dividends, then $F(t,\tau) = V(t)/P(t,\tau)$. By contrast, for the forward measure we have

$$E_t^{\mathcal{Q}_\tau}[P(t,\tau)\,(Y(\tau) - F(t,\tau)] = 0, \tag{4.8}$$

which produces

$$F(t,\tau) = E_t^{\mathcal{Q}_\tau}[Y(\tau)]. \tag{4.9}$$

Comparing either (4.9) and (4.4) or (4.7) and (4.1), we see that forward prices can be expressed as the ratio of two asset values. Also note that even though the *representations* for $F(t,\tau)$ given in (4.7) and (4.9) in terms of the expectations are different, the *values* for $F(t,\tau)$ are identically equal.

Contrast the forward price with the *futures price* $\mathcal{F}(t,\tau)$. Under the futures measure we have[6]

$$\mathcal{F}(t,\tau) = E_t^{\mathcal{Q}_\beta}[Y(\tau)], \tag{4.10}$$

[6] See Duffie (1996, chapter 8).

while under the forward measure we have, by (3.7),

$$\begin{aligned}\mathcal{F}(t,\tau) &= E_t^{\mathcal{Q}_\beta}[Y(\tau)] \\ &= \frac{E_t^{\mathcal{Q}_\tau}\left[\big(\beta(\tau)/P(\tau,\tau)\big)\,Y(\tau)\right]}{\big(\beta(t)/P(t,\tau)\big)} \\ &= P(t,\tau)\,E_t^{\mathcal{Q}_\tau}\left[\exp\left(\int_t^\tau r(t)\,du\right)Y(\tau)\right].\end{aligned} \tag{4.11}$$

Note that unless interest rates are deterministic, futures prices bear no simple relationship to asset values.

5. Markovian state-variable representations

The futures measure and the Feynman–Kac solution. In this section, we suppose the economy can be characterized by a finite number of state variables $X(t)$. Thus we will write $V(t) = V(X(t),t)$, $r(t) = r(X(t))$, and $\lambda(t) = \lambda(X(t))$, where $V(x,t)$, $r(x)$, and $\lambda(x)$ are functions of x. The absence of arbitrage is characterized by (2.6), which—in a state-variable setting—we can write as

$$\bar{\mu}_V(X(t),t) - \lambda(X(t))^\top \bar{\sigma}_V(X(t),t) = r(X(t))\,V(X(t),t). \tag{5.1}$$

Let dynamics of $X(t)$ be given by

$$dX(t) = \bar{\mu}_X(X(t))\,dt + \bar{\sigma}_X(X(t))^\top dW(t). \tag{5.2}$$

Then Ito's lemma gives

$$dV(t) = \bar{\mu}_V(X(t),t)\,dt + \bar{\sigma}_V(X(t),t)^\top dW(t),$$

where

$$\bar{\mu}_V(x,t) = \bar{\mu}_X(x)^\top V_x(x,t) + \frac{1}{2}\,\mathrm{tr}\left[V_{xx}(x,t)\,\bar{\sigma}_X(x)^\top\bar{\sigma}_X(x)\right] - V_t(x,t) \tag{5.3a}$$

and

$$\bar{\sigma}_V(x,t) = \bar{\sigma}_X(x)\,V_x(x,t), \tag{5.3b}$$

where $\mathrm{tr}[a]$ is the trace of matrix a, and where

$$V_t(x,t) := \frac{\partial V(x,t)}{\partial t},$$

$$V_x(x,t) := \begin{pmatrix} \frac{\partial V(x,t)}{\partial x_1} \\ \vdots \\ \frac{\partial V(x,t)}{\partial x_d} \end{pmatrix}, \quad \text{and} \quad V_{xx}(x,t) := \begin{pmatrix} \frac{\partial^2 V(x,t)}{\partial x_1^2} & \cdots & \frac{\partial^2 V(x,t)}{\partial x_1\,\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V(x,t)}{\partial x_d\,\partial x_1} & \cdots & \frac{\partial^2 V(x,t)}{\partial x_d^2} \end{pmatrix}.$$

Using (5.3), we can write the absence-of-arbitrage condition (5.1) as a partial differential equation:

$$\hat{\bar{\mu}}_X(x)^\top V_x(x,t) + \frac{1}{2}\,\mathrm{tr}\left[V_{xx}(x,t)\,\bar{\sigma}_X(x)^\top\bar{\sigma}_X(x)\right] - V_t(x,t) \;=\; r(x)\,V(x,t), \tag{5.4}$$

where

$$\hat{\hat{\mu}}_X(x) := \bar{\mu}_X(x) - \bar{\sigma}_X(x)^\top \lambda(x). \tag{5.5}$$

Note that if $X(t)$ is a vector of asset values (for assets that pay no dividends), then by (2.6) $\hat{\hat{\mu}}_X(X(t)) = r(X(t))\, X(t)$, in which case (5.4) is a *preference-free* characterization of $V(x,t)$—it does not depend on the market price of risk λ.

We can obtain equation (5.4) from a different measure. In particular, if the dynamics for the state-variables were given by

$$dX(t) = \hat{\hat{\mu}}_X(X(t))\, dt + \bar{\sigma}_X(X(t))^\top d\widehat{W}(t), \tag{5.6}$$

where

$$d\widehat{W}(t) = dW(t) + \lambda(t)\, dt.$$

The Feynman–Kac solution to (5.4) is given by

$$V(x,t) = \widehat{E}_t\left[\exp\left(-\int_t^s r(X(u))\, du\right) V(X(s),s)\right], \tag{5.7}$$

where $\widehat{E}_t$ indicates that X is assumed to solve (5.6).[7] Note that the Feynman–Kac solution technique is formally identical to valuing securities using the futures measure.

The the Feynman–Kac solution (5.7) involves the entire path of the short rate, which is not, in general, representable as a function of the state variables X.[8] As such, as long as interest rates are not deterministic, it is only really a solution in the sense that it provides a guide to Monte Carlo techniques.

The induced distributions of state-variables. In this section, we show how to use forward and futures prices to uncover the distribution of a state variable under the futures and forward equivalent martingale measures at some fixed time τ in the future.

For simplicity, we will consider a single state variable. Consider the following payoff at time τ: $\max\{X(\tau) - K, 0\}$, where X is the single state variable we are interested in.[9] The expectation of this payoff can be expressed in terms of the distributions on $X(\tau)$ induced by the respective equivalent martingale measures:

$$E_t^i[\max\{X(\tau) - K, 0\}] = \int_K^\infty (x - K)\, f^i(x)\, dx, \tag{5.8}$$

where $i \in \{\mathcal{Q}_\tau, \mathcal{Q}_\beta\}$ and $f^i(x)$ is the induced (conditional) density of $X^i(\tau)$ under i. Differentiating both sides of (5.8) twice with respect to K produces

$$\frac{\partial^2}{\partial K^2}\, E_t^i[\max\{X(\tau) - K, 0\}] = -f^i(K). \tag{5.9}$$

[7] See Duffie (1996, chapter 5 and appendix E).

[8] One can, of course, always expand the set of state-variables (to the set of *paths*, for example) to obtain such a representation. But this state-variable setting would lose its structural appeal.

[9] The payoff $\max\{K - X(\tau), 0\}$ would work just as well.

Recall the representations (4.9) and (4.10) for the forward and futures prices, respectively, and let $Y(\tau) = \max\{X(\tau) - K, 0\}$. Then we using (5.9) we can write

$$-\frac{\partial^2}{\partial K^2} F(t,\tau) = f^{\mathcal{Q}_\tau}(K) \tag{5.10a}$$

and

$$-\frac{\partial^2}{\partial K^2} \mathcal{F}(t,\tau) = f^{\mathcal{Q}_\beta}(K) \tag{5.10b}$$

In other words, forward prices on the payoff $\max\{K - X(\tau), 0\}$ can be used to reveal the distribution of $X(\tau)$ induced by the forward measure, while futures prices on the same payoff can be used to reveal the distribution induced by the futures measure.

Let $C(X(t), K)$ be the value of a European call option on X that expires at time τ with strike price K. Note that $C(X(\tau), K) = \max\{K - X(\tau), 0\}$. Thus we can interpreted $F(t,\tau)$ and $\mathcal{F}(t,\tau)$ as forwards and futures on options that expire on the delivery date. Moreover, recall that the value of an asset with a given payoff is closely related the futures price associated with that payoff. In particular, applying (4.4), we have

$$C(X(t), K) = P(t,\tau)\, E_t^{\mathcal{Q}_\tau}\left[C(X(\tau), K)\right]. \tag{5.11}$$

Therefore, European options prices can be used to reveal the distribution of $X(\tau)$ induced by the forward measure:

$$-\frac{1}{P(t,\tau)}\frac{\partial^2}{\partial K^2} C(X(t), K) = f^{\mathcal{Q}_\tau}(K).$$

There is, however, no corresponding simple relationship between the futures price of a payoff and asset value of the payoff that allows us to use options prices directly to uncover the distribution of $X(\tau)$ induced by the futures measure (except when the two measures coincide).[10]

6. Risk-neutrality and random rates

We wish to consider economies where interest rates are random. The first point to make is that it is not obvious how "risk neutral" investors determine asset prices in general. With regard to this issue, Cox, Ingersoll, Jr., and Ross (1981) note, among other things, that in the most natural case risk neutrality itself implies that interest rates are *deterministic*.

For example, consider an economy where at any time t consumers maximize the expected utility of their consumption plan, given by

$$E_t^{\mathcal{P}}\left[\int_t^T u(c(s), s)\, ds\right].$$

The term "risk-neutral" often applies to investors whose utility function is linear in consumption: $u(c(t), t) = e^{-\rho t}\, h(t)\, c(t)$, where $h(t)$ is strictly positive. In this

[10]Should we say something about how the the forward measure is closely related to the Green's function, which is also known as the fundamental solution to the PDE. . . . ?

economy, the state-price deflator is given by the marginal utility of wealth:[11] $m(t) = u_c(c(t), t) = e^{-\rho\, t}\, h(t)$. Given the positivity of $h(t)$, we can write

$$\frac{dm(t)}{m(t)} = \big(\mu_h(t) - \rho\big)\, dt + \sigma_h(t)^\top dW(t). \tag{6.1}$$

where $\mu_h(t)$ and $\sigma_h(t)$ are the drift and volatility of $dh(t)/h(t)$. By inspection of (6.1), we see that $r(t) = \rho - \mu_h(t)$ and $\lambda(t) = -\sigma_h(t)$.

Clearly, the implications of risk-neutrality (as characterized in this setting) depend on the dynamic properties of h. The typical assumption is that h is constant, in which case the interest rate is constant and the market price of risk is zero. This is the case that Cox, Ingersoll, Jr., and Ross (1981) discussed. This is also the setting in which Cox and Ross (1976) characterized risk-neutral pricing. By contrast, if h is not predictable (in other words, if $\sigma_h(t) \not\equiv 0$), there is a non-zero market price of risk in this economy of so-called risk-neutral investors.

The case of most interest to us at this point is when h is predictable but its drift is random, so that we have random interest rates combined with no risk premiums. This would seem to characterize risk-neutrality when interest rates are random: The expected return on all assets equals the risk-free rate. The state-price deflator–asset in this case is the money-market account: $m(t)^{-1} = \beta(t)$. Note, however, that a simple change of numeraire can destroy this feature. In other words, if there is no risk premium in real terms, there will be one in nominal terms as long as the price level is not deterministic. Conversely, if there were no risk premium in nominal terms, it would be unlikely that there would be no risk premium in real terms, and our link to preferences would be broken.

The forward measure as physical measure. As we have seen, the bond price $P(t, \tau)$ is a perfectly valid deflator–asset, and is associated with an equivalent martingale measure. It is natural to wonder if it is possible to construct an economy in which $\mathcal{Q}_\tau$ equals the physical measure $\mathcal{P}$. The answer is yes: suppose that the representative investor has a rigid horizon of τ, does not consume at any other time and tries to maximize expected wealth at τ. The yardstick to measure the worth of any investment prospect is the expected contribution to aggregate wealth at time τ.

With the tools developed above, it is easy to show that for CIR's economy of risk-neutral investors who consume only at a single point of time in the future, $\mathcal{P} = \mathcal{Q}_\tau$. In particular, note that CIR show that the marginal utility of wealth ($Q(Y, t)$ in their notation) equals the right-hand side of (4.5) (see their equation (37)). Therefore, the state-price deflator for that economy is the inverse of the zero coupon bond that matures at time τ.

Note that an investment at time t that delivers a random payoff $g(s)$ at time $s \le \tau$ is not worthless. Instead, its value $V(t)$ is[12]

$$V(t) = P(t, \tau)\, E_t^{\mathcal{P}}\left[\frac{g(s)}{P(s, \tau)}\right].$$

[11] See Duffie (1996, chapter 10).
[12] *cf.* (4.3).

Note the use the physical measure to calculate expectation since the assumption is that it is the same as $\mathcal{Q}_\tau$. This pricing formula has the natural interpretation that the investor plans to reinvest the proceeds into the bond that matures at τ.[13]

Since the investor maximizes expected wealth, there is a good reason to call him risk neutral. Therefore, a pricing system under which $\mathcal{Q}_\tau = \mathcal{P}$ can reasonably be called risk neutral. But note that under this definition, assets do not have equal expected rates of return. As we have seen, the expected return offers a compensation for risk, where the price of risk is the proportional volatility, $\sigma_P(t,\tau)$, of the deflator–asset $P(t,\tau)$.

Note that changing the numeraire does not affect the salient features of this economy: The representative investor's rigid horizon does not depend on whether asset prices are measured in real terms or nominal terms. Thus this notion of risk-neutral pricing is as robust as it is unappealing.

Heath, Jarrow, and Morton (1992, HJM) derive the absence-of-arbitrage restriction in terms of forward rates. Let the process for forward rates be given by

$$df(t,T) = \mu_f(t,T)\,dt + \sigma_f(t,T)^\top dW(t).$$

The HJM absence-of-arbitrage restriction is

$$\mu_f(t,T) = \sigma_f(t,T)^\top \left(\lambda(t) + \int_{s=t}^{T} \sigma_f(t,s)\,ds\right),$$

for all $T \geq t$. In this economy, $\lambda(t) = \sigma_P(t,\tau) = -\int_{s=t}^{\tau} \sigma_f(t,\tau)\,ds$, and thus

$$\mu_f(t,T) = -\sigma_f(t,T)^\top \int_{s=T}^{\tau} \sigma_f(t,s)\,ds. \tag{6.2}$$

We see from (6.2) that forward rates for time τ are martingales and therefore are unbiased predictors of the spot rate at time τ. In other words, when the forward measure is the physical measure, the unbiased expectations hypothesis holds for time τ only. On the other hand, (4.5) is the representation for bond prices under the return-to-maturity hypothesis.[14]

7. An empirical investigation: The term structure

We use multi-factor CIR models of the term structure to investigate the differences among the measures with respect to the short-term interest rate.

The distributions in general. In this class of models, the instantaneous risk-free rate equals the sum of unobserved, independent factors (or state-variables), $r(t) = \sum_{j=1}^{d} x_j(t)$. We are interested in the distribution of $r(s)$ conditional on the $x_j(t)$ for $s \geq t$. Under each of the three measures, the distribution of $x_j(s)$ conditional on $x_j(t)$ is $\delta_j\,\chi(\nu_j,\eta_j)$ where δ_j is a constant and where $\chi(\nu_j,\eta_j)$ is a non-central chi-square variate with ν_j degrees of freedom and non-centrality parameter η_j.

[13]We have so far not considered payoffs that occur beyond τ. Investors will of course want to sell at τ, and attach values to them.

[14]Of course, the two hypotheses cannot hold for all maturities simultaneously.

The conditional distribution of the sum $r(s) = \sum_{i=1}^{d} x_i(s)$ can be obtained as follows: The characteristic function of the sum of independent random variables is the product of the univariate characteristic functions. As Chen and Scott (1995) show, the characteristic function for $\delta_j\,\chi(\nu_j, \eta_j)$ is given by

$$\phi_j(u) = (1 - 2\,i\,u\,\delta_j)^{-\nu_j/2}\,\exp\left\{\frac{i\,\eta_j\,u\,\delta_j}{1 - 2\,i\,u\,\delta_j}\right\},$$

where $i = \sqrt{-1}$. The characteristic function for $r(s)$ then is given by $\Phi(u) = \prod_{j=1}^{d} \phi_j(u)$. The distribution for $r(s)$ is given by

$$\begin{aligned} f(r(s) \mid x_1(t), \ldots, x_d(t)) &= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{-i\,r(s)\,u}\,\Phi(u)\,du \\ &= \frac{2}{\pi} \int_0^{\infty} \cos(r(s)\,u)\,\Re(\Phi(u))\,du, \end{aligned}$$

where $\Re(\Phi(u))$ denotes the real part of $\Phi(u)$. The second equality relies on the non-negativity of the x_j.[15]

The futures measure. The dynamics of the factors under $\mathcal{Q}_\beta$ is given by

$$dx_j(t) = \big(a_j + b_j\,x_j(t)\big)\,dt + c_j\,\sqrt{x_j(t)}\,dW_j^{\mathcal{Q}_\beta}(t),$$

where all of the Brownians are independent. Cox, Ingersoll, Jr., and Ross (1985) showed that conditional on $x_j(t)$, for $\tau \geq t$, $x_j(\tau)$ is distributed as $\delta_j^\beta\,\chi(\nu_j, \eta_j^\beta)$, where $\chi(\nu_j, \eta_j^\beta)$ is a non-central chi-square variate with ν_j degrees of freedom and non-centrality parameter η_j^β, and where

$$\nu_j = \frac{4\,a_j}{c_j^2}, \quad \eta_j^\beta = \frac{4\,b_j\,e^{b_j\,(\tau-t)}}{c_j^2\,\left(e^{b_j\,(\tau-t)} - 1\right)}\,x_j(t), \quad \text{and} \quad \delta_j^\beta = \frac{c_j^2\,\left(e^{b_j\,(\tau-t)} - 1\right)}{4\,b_j}.$$

The physical measure. Now suppose the market price of risk is given by $\lambda_j(t) = (d_j/c_j)\,\sqrt{x_j(t)}$. Then the process for the short rate under the physical measure is given by

$$dx_j(t) = \big(a_j + (b_j - d_j)\,x_j(t)\big)\,dt + c_j\,\sqrt{x_j(t)}\,dW(t).$$

The distribution for the short rate under the physical measure is the same as under the futures measure, but with b_j replaced by $b_j - d_j$.

The forward measure. To find the distribution for the short rate under the forward measure in this economy, first note that zero-coupon bond prices are given by

$$P(x, t, s) = \exp\Big\{ - A(s-t) - \sum_{j=1}^{d} B_j(s-t)\,x_j \Big\}, \tag{7.1}$$

[15]See Chen and Scott (1995) on this point.

where the factor loadings, $B_j(s-t)$, are given by

$$B_j(s-t) = \frac{2\,(e^{\gamma_j\,(s-t)} - 1)}{(\gamma_j + b_j)(e^{\gamma_j\,(s-t)} - 1) + 2\,\gamma_j} \quad \text{and where} \quad \gamma_j = \sqrt{b_j + 2\,c_j^2}.$$

Equation (7.1) is the standard generalization of the one-factor bond pricing formula in Cox, Ingersoll, Jr., and Ross (1985).

The relative volatility of the zero-coupon bond that matures at time τ is given by

$$\sigma_P(t, \tau - t) = \begin{pmatrix} -c_1\,B_1(\tau - t)\,\sqrt{x_1(t)} \\ \vdots \\ -c_d\,B_d(\tau - t)\,\sqrt{x_d(t)} \end{pmatrix}.$$

Therefore, the change of in the drift of $x_j(t)$ induced by the change of measure from $\mathcal{Q}_\beta$ to $\mathcal{Q}_\tau$ is

$$-\sigma_P(t, \tau - t)^\top \sigma_X(t) = c_j^2\,B_j(\tau - t)\,x_j(t),$$

so that the process for $x_j(t)$ under $\mathcal{Q}_\tau$ is

$$dx_j(t) = \Big(a_j + \big(b_j - c_j^2\,B_j(\tau - t)\big)x_j(t)\Big)\,dt + c_j\,\sqrt{x_j(t)}\,dW^{\mathcal{Q}_\tau}(t). \tag{7.2}$$

Chen and Scott (1995) show that conditional on $x_j(t)$, $x_j(\tau)$ is distributed under $\mathcal{Q}_\tau$ as $\delta_j^\tau\,\chi^2(\nu_j, \eta_j^\tau)$, where ν_j is given above and

$$\eta_j^\tau = \frac{2\,\varphi_j^2\,e^{\gamma_j\,(\tau-t)}}{\varphi_j + \psi_j}\,x_j(t) \quad \text{and} \quad \delta_j^\tau = 2\,(\varphi_j + \psi_j),$$

where

$$\varphi_j = \frac{2\,\gamma_j}{c_j^2\,(e^{\gamma_j\,(\tau-t)} - 1)} \quad \text{and} \quad \psi_j = \frac{\gamma_j - b_j}{c_j^2}.$$

Results. Figures 1–6 show the results. We took the parameters that are fit by Chen and Scott (1993) for one-, two-, and three-factor CIR models using the McCulloch data from the mid-1960s to 1987. The parameter values are shown in Table 1. The figures show PDFs and CDFs for one-, five-, ten-, and thirty-year horizons. The solid lines indicate $\mathcal{Q}_\tau$, the long dashes indicate $\mathcal{P}$, and the short dashes indicate $\mathcal{Q}_\beta$. The conditional distributions are evaluated at the unconditional means under the physical measure as implied by the models, $\bar{x}_j$. The three models put the mean short rate in the neighborhood of five to six percent.

The results for the one-factor model are different from those of the two- and three-factor models. For the one-factor model, $\mathcal{Q}_\beta$ and $\mathcal{Q}_\tau$ are essentially the same at all horizons up to thirty years. The two equivalent martingale measures diverge somewhat from the physical measure by the five-year horizon, but the divergence does not increase beyond that. It has been widely documented that one-factor models do a poor job of matching both the time-series and cross-sectional properties of the term structure. Thus, there is reason to be skeptical about these results.

Now consider the results for the the two- and three-factor models. At the one-year horizon, $\mathcal{Q}_\beta$ and $\mathcal{Q}_\tau$ are virtually indistinguishable from each and only moderately

a	b	c	d	$\bar{x} = \frac{-a}{b+d}$
0.029	−0.424	0.082	−0.045	0.062
0.025	−0.647	0.131	−0.119	0.032
0.00002	0.041	0.053	−0.042	0.021
0.053	−1.601	0.137	−0.032	0.032
0.00005	0.148	0.075	−0.153	0.011
0.00006	0.131	0.184	−0.137	0.009

Table 1. Estimated coefficients for one-, two-, and three-factor CIR models

different from $\mathcal{P}$. At the five-year horizon, the difference between the physical measure and the two martingale measures is quite distinct, while the martingale measures are still close to each other. At the ten-year horizon, the difference between $\mathcal{P}$ and $\mathcal{Q}_\beta$ is striking, and $\mathcal{Q}_\tau$ has moved noticeably toward $\mathcal{P}$. By the thirty-year horizon, $\mathcal{P}$ and $\mathcal{Q}_\tau$ are quite close, while $\mathcal{Q}_\beta$ is quite far away.

8. Conclusion

We have described two equivalent martingale measures, the futures measure (associated with the money-market account) and the forward measure (associated with a zero-coupon bond). Both measures are useful in representing asset values. Under the futures measure, the expected return on assets equals the risk-free rate. This feature, however, has only a tenuous connection with risk-neutrality per se, especially when returns are measured in nominal terms. Under the forward measure asset values equal the present value of a certainty equivalent. This feature can be associated with risk-neutrality, but of an unappealing sort. Moreover, unless interest rates are deterministic, the two measures are different, and therefore one cannot have both features in an economy where interest rates are random. However, as our empirical investigation revealed, for horizons up to a year at least, the difference between the two equivalent martingale measures appears to be negligible. On the other hand, for longer horizons (in the multi-factor models) the two measures diverge markedly, and the forward measure tends to merge with the physical measure.

Appendix A. Deflators that produce equivalent martingale measures

In the body of the paper we showed that an equivalent martingale measure can be constructed using a deflator that is the inverse of a strictly positive asset value. In this Appendix we show that the inverse of any deflator that yields an equivalent martingale must have the dynamics of an asset, thereby establishing the necessity noted in the body.

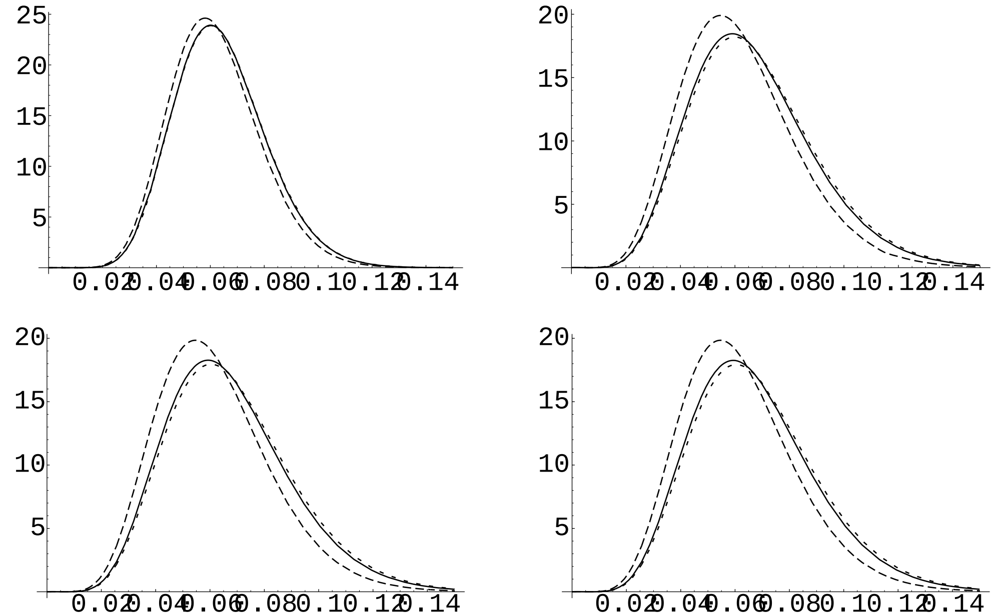

FIGURE 1. One-factor CIR model: PDFs

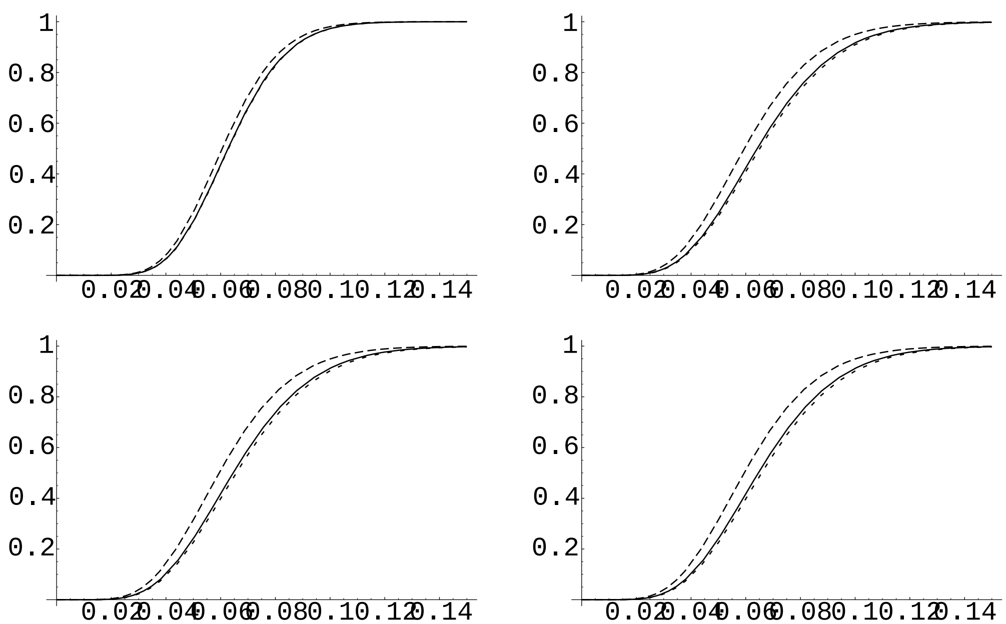

FIGURE 2. One-factor CIR model: CDFs

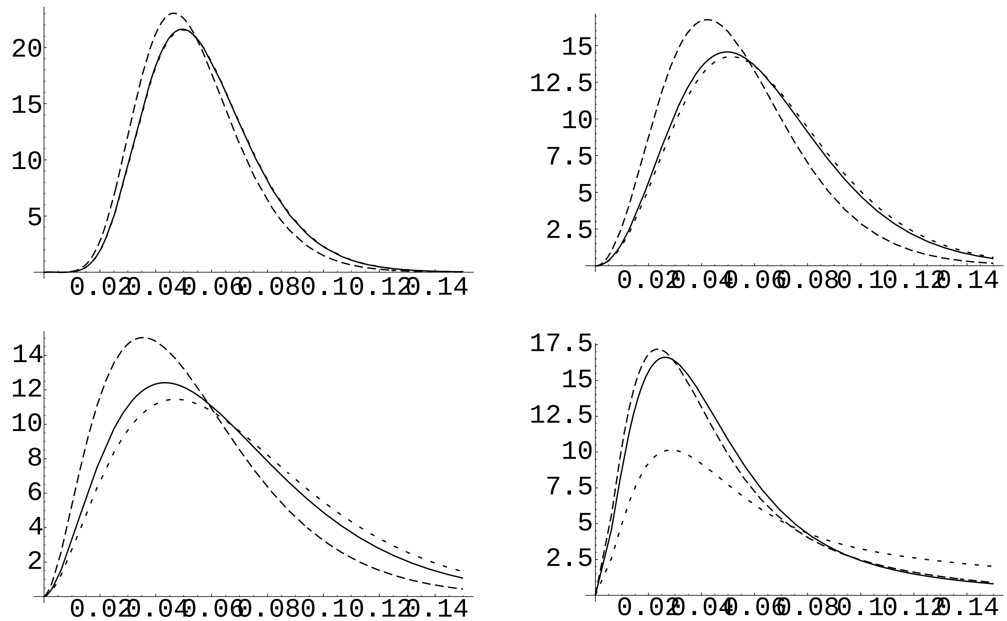

FIGURE 3. Two-factor CIR model: PDFs

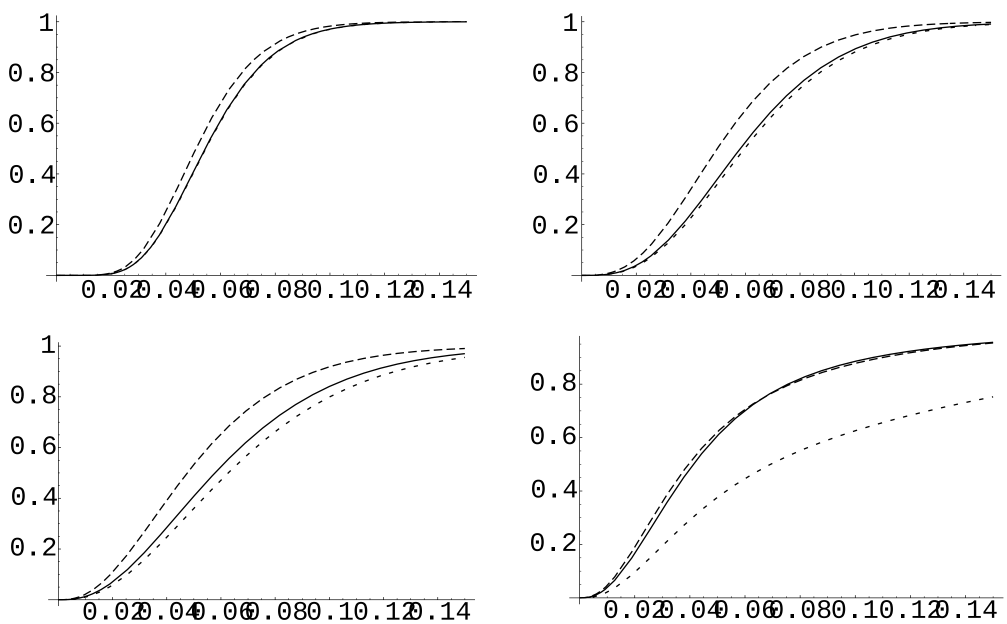

FIGURE 4. Two-factor CIR model: CDFs

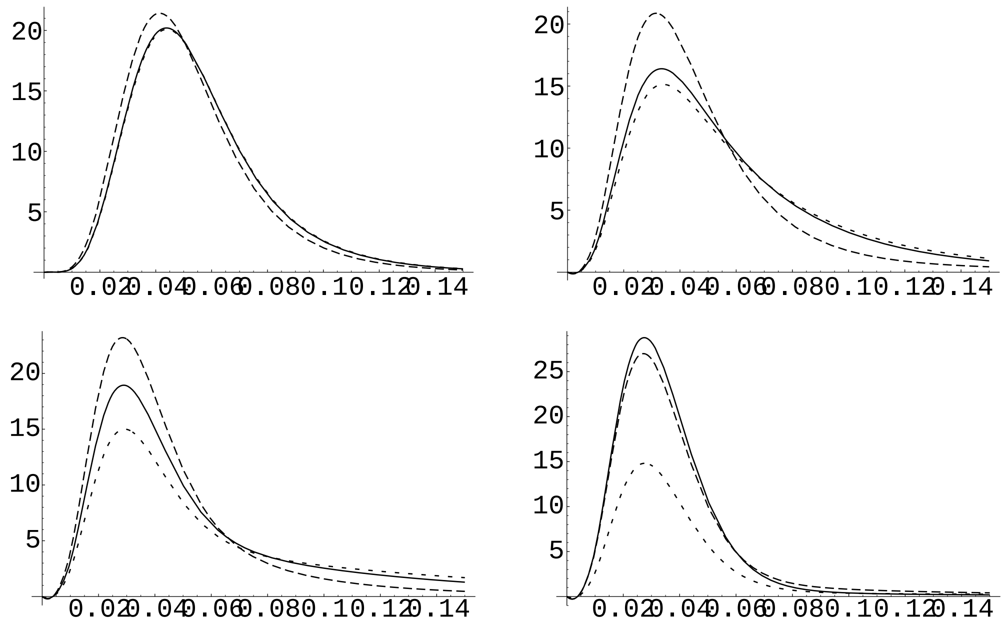

FIGURE 5. Three-factor CIR model: PDFs

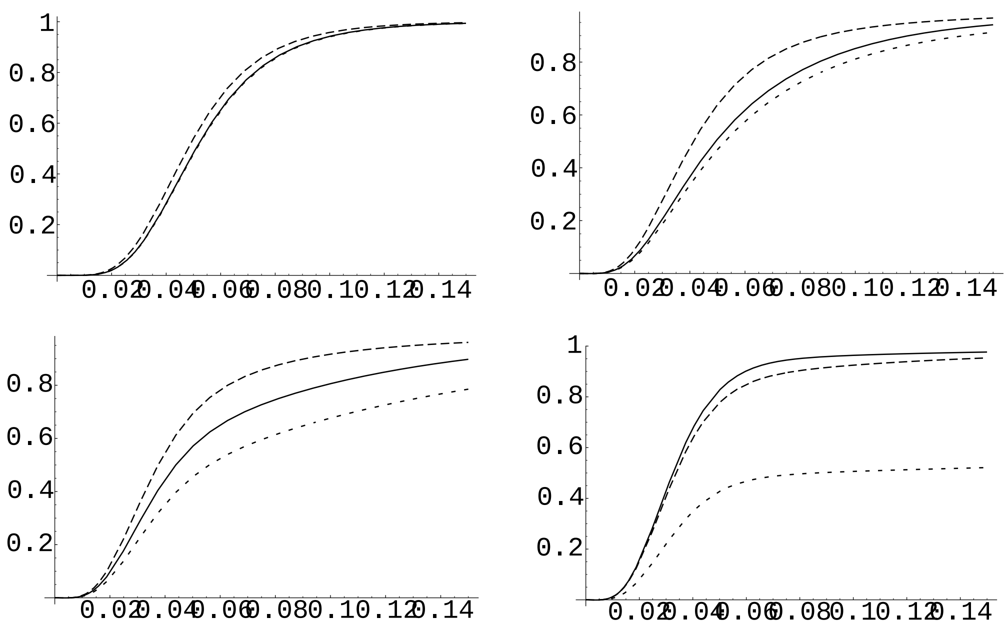

FIGURE 6. Three-factor CIR model: CDFs

Let $Y(t)$ be a strictly positive process, so that we can write

$$\frac{dY(t)}{Y(t)} = \mu_Y(t)\, dt + \sigma_Y(t)^\top dW(t).$$

Let the process for the value of an asset, $V(t)$, be given by (2.3) and (2.6), and let $\xi_Y(t) := Y(t)\, V(t)$ be the deflated value of an asset. The process for $\xi_Y(t)$ is given by

$$d\xi_Y(t) = \mu_{\xi_Y}(t)\, dt + \sigma_{\xi_Y}(t)^\top dW(t),$$

where

$$\mu_{\xi_Y}(t) = Y(t) \left\{ V(t) \big(r(t) + \mu_Y(t) \big) + \bar{\sigma}_V(t)^\top \big(\lambda(t) + \sigma_Y(t) \big) \right\}$$

and

$$\sigma_{\xi_Y}(t) = Y(t) \left\{ \bar{\sigma}_V(t) + V(t)\, \sigma_Y(t) \right\}.$$

Under an equivalent measure, the process for $\xi_Y(t)$ is given by

$$d\xi_Y(t) = \big(\mu_{\xi_Y}(t) - \theta(t)^\top \sigma_{\xi_Y}(t) \big)\, dt + \sigma_{\xi_Y}(t)^\top dW^Y(t).$$

The question is this: Under what conditions is $\xi_Y(t)$ a martingale for an arbitrary asset $V(t)$? In other words, we seek the conditions under which

$$\mu_{\xi_Y}(t) - \theta(t)^\top \sigma_{\xi_Y}(t) = 0 \tag{A.1}$$

for arbitrary $\bar{\sigma}_V(t)$ and $V(t)$. Rearranging (A.1) produces

$$\bar{\sigma}_V(t)^\top \left\{ \lambda(t) + \sigma_Y(t) - \theta(t) \right\} + V(t) \left\{ r(t) + \mu_Y(t) - \theta(t)^\top \sigma_Y(t) \right\} = 0. \tag{A.2}$$

The first term on the left-hand side of (A.2) implies $\theta(t) = \lambda(t) + \sigma_Y(t)$. Using this expression for $\theta(t)$, the second term implies

$$\mu_Y(t) = -r(t) + \lambda(t)^\top \sigma_Y(t) + \|\sigma_Y(t)\|^2. \tag{A.3}$$

Using (A.3), we can write the process for $U(t) := Y(t)^{-1}$ as

$$\frac{dU(t)}{U(t)} = \big(r(t) + \lambda(t)^\top \sigma_U(t) \big)\, dt + \sigma_U(t)^\top dW(t), \tag{A.4}$$

where $\sigma_U(t) = -\sigma_Y(t)$. Thus (A.4) shows that the inverse of $Y(t)$ has the dynamics of an asset value.

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