THE ABSENCE OF ARBITRAGE AND GENERAL EQUILIBRIUM IN CONTINUOUS TIME: AN OVERVIEW

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Abstract. We review the absence-of-arbitrage restrictions in a Brownian motion setting, relying on the existence of a *state–price deflator*. We show how to find useful representations by either changing the numeraire or by changing the *deflator–asset* and its associated equivalent martingale measure. We examine the relation between utility and the state–price deflator, and the absence-of-arbitrage condition for a perpetuity is applied to finding (i) the optimal relationship between consumption and wealth and (ii) the nominal interest rate.

1. Introduction

Absence-of-arbitrage restrictions determine asset prices by assuming "you can't get something for nothing." An *arbitrage* is a trading strategy that (i) requires no inflows from the trader yet (ii) produces outflows for the trader with positive probability. The central proposition of arbitrage-free asset pricing is this: If it is possible to choose the units in which trading gains are measured in such a way as to make all gains unpredictable (*i.e.*, make them martingales), then there are no arbitrage opportunities. When asset values are measured in these units, they are referred to as *deflated* asset values, and the deflator that does the trick is known as the *state–price deflator*.

The rate of change of the state–price deflator can be decomposed into two parts—expected and unexpected—each of which has a particularly useful interpretation. The expected rate of change is (the negative of) the short-term risk-free interest rate. The unexpected rate of change is (the negative of) the price of risk: It determines the covariance between the rate of change of the state–price deflator and the rate of return on an asset. This covariance is (the negative of) the risk premium for the asset. From a modeling perspective, this decomposition is convenient because it

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allows one to model the interest rate and the price of risk directly and independently (subject only to the existence of the state–price deflator itself).

Once we have identified the state–price deflator in terms of US dollars, say, it is straightforward to find the state–price deflator for gains measured in French francs by using the foreign exchange rate to revalue assets, their dividends, and (the inverse of) the state–price deflator itself. The expected rate of change of the new state–price deflator is (the negative of) the interest rate measured in francs and the unexpected rate of change is (the negative of) the price of risk measured in francs. The same technique can be used to change from real values to nominal values (and vice-versa) using the price level. More generally, one can define a new numeraire in terms of *any* positive process, for example the value of a strictly positive asset. In such a case, it turns out that the value of this asset deflated by the original state–price deflator is the *new* state–price deflator. As such, its expected rate of change is (the negative of) the interest rate and its unexpected rate of change is (the negative of) the price of risk, both measured in terms of the new numeraire.

In general equilibrium, asset prices are determined by "marginal cost equals marginal benefit," where the marginal cost is the marginal utility of consumption foregone by the purchase of the asset, and the marginal benefit is the conditional expectation of the marginal utility (in present value terms) to be obtained from the sale of the asset (and from any dividends the asset may pay). In other words, asset values deflated by (the present value) of marginal utility are martingales. Apparently, then, in general equilibrium the marginal utility of consumption adjusted for discounting (the utility gradient) *is* the state–price deflator.

A general-equilibrium valuation formula for asset prices can be obtained directly from the utility function and the dynamics of equilibrium consumption without reference to the optimal relationship between consumption and wealth. The optimal relation between consumption and wealth can be uncovered by treating wealth as an asset and applying the valuation equation. This approach differs from that of Cox, Ingersoll, Jr., and Ross (1985a), in which the optimal relationship between consumption and wealth is determined first, and then their valuation formula is written in terms of that optimal relationship.

Duffie and Skiadas (1994) demonstrate that "the first-order conditions for optimality of an agent maximizing a 'smooth' utility can be formulated as the martingale property of prices, after normalization by a 'state–price' process." Moreover, they show how to compute the state–price deflator for "a wide class of dynamics utilities." Using these results and the absence-of-arbitrage representations, we show how to solve infinitely-lived representative agent models of general equilibrium without solving differential equations. The central insight is that given the price process for an asset, the absence of arbitrage determines the dividend process. Turning this around, one can model the (state–price) deflated asset value, where the expected return is (minus) the dividend rate, and the absence of arbitrage determines the capital gain process.

In general equilibrium, the capital stock and currency are the two central assets. With homothetic preferences and linear technology, there is sufficient homogeneity in these models so that for the most part one need model only the dynamics of the economy. However, the equilibrium *dynamics* depend on the *level* of the consumption–capital ratio and the nominal interest rate, which are the dividends of the capital stock and currency, respectively. If one directly models these dividends, then their dynamics follow. Otherwise these dividends and their dynamics are the solutions to differential equations.

Section 2 presents two main techniques for finding useful representations for asset prices in an arbitrage-free framework: (i) changing the numeraire and (ii) changing the deflator–asset and its associated martingale measure. It turns out the the first technique is more the important of the two for solving general equilibrium models. Section 3 examines the relationship between utility and the state–price deflator, and the perpetuity equation that is developed in Section 2 is applied to finding (i) the optimal relationship between consumption and wealth and (ii) the nominal interest rate.

2. Absence of arbitrage

The stochastic setting that we adopt is characterized by a vector of d independent Brownian motions, W.[1] In this setting, a *deflator* is a positive Ito process whose dynamics can be written

$$\frac{dY(t)}{Y(t)} = \mu_Y(t)\,dt + \sigma_Y(t)^\top dW(t), \tag{2.1}$$

where "$^\top$" denotes the transpose. In general, the drift and diffusion of this and other Ito processes may be Ito processes themselves. Let the value of an asset V (which need not be positive) be an Ito process:

$$dV(t) = \bar{\mu}_V(t)\,dt + \bar{\sigma}_V(t)^\top dW(t). \tag{2.2}$$

Let the *cumulative dividend* of the asset D also be an Ito process:

$$dD(t) = \bar{\mu}_D(t)\,dt + \bar{\sigma}_D(t)^\top dW(t). \tag{2.3}$$

If V is a *positive asset*, we may write

$$\begin{aligned}\frac{dV(t)}{V(t)} &= \mu_V(t)\,dt + \sigma_V(t)^\top dW(t)\\ \frac{dD(t)}{V(t)} &= \delta(t)\,dt + \sigma_D(t)^\top dW(t).\end{aligned}$$

where

$$\mu_V(t) := \frac{\bar{\mu}_V(t)}{V(t)}, \quad \sigma_V(t) := \frac{\bar{\sigma}_V(t)}{V(t)}, \quad \delta(t) := \frac{\bar{\mu}_D(t)}{V(t)}, \quad \text{and} \quad \sigma_D(t) := \frac{\bar{\sigma}_D(t)}{V(t)}.$$

Define the *gain* to be the sum of the asset's value and its cumulative dividend: $G := V + D$. For a given deflator Y, the dynamics of the *deflated gain* are given by

$$dG^Y(t) = dV^Y(t) + dD^Y(t),$$

[1]See Duffie (1996, Chapter 6) for omitted details.

where the deflated value of the asset is defined by

$$V^Y(t) := Y(t)\,V(t) \tag{2.4a}$$

and the deflated cumulative dividend is defined by

$$D^Y(0) := Y(0)\,D(0) \tag{2.4b}$$

$$\begin{aligned} dD^Y(t) &:= Y(t)\,dD(t) + Y(t)\,\sigma_Y(t)^\top \bar{\sigma}_D(t)\,dt \\ &= Y(t)\left\{\bar{\mu}_D(t) + \sigma_Y(t)^\top \bar{\sigma}_D(t)\right\}dt + Y(t)\,\bar{\sigma}_D(t)^\top dW(t) \\ &= \mu_D^Y(t)\,dt + \sigma_D^Y(t)^\top dW(t). \end{aligned} \tag{2.4c}$$

The apparent asymmetry in the definitions of V^Y and D^Y is required to produce the necessary symmetry in the absence-of-arbitrage condition.

A *trading strategy* θ is a vector-valued Ito process that represents the positions (long or short) in each security. The total gain represented by θ is $\int \theta(s)^\top dG(s)$, where $G = V + D$ is understood to be a vector of gains here. A trading strategy is *self-financing* if

$$\theta(T)^\top V(T) = \theta(t)^\top V(t) + \int_{s=t}^{T} \theta(s)^\top dG(s).$$

A self-financing strategy is an *arbitrage* with respect to G if for some t and T

$$\theta(t)^\top V(t) < 0 \quad \text{and} \quad \theta(T)^\top V(T) \geq 0$$

or

$$\theta(t)^\top V(t) \leq 0 \quad \text{and} \quad \theta(T)^\top V(T) > 0.$$

By the *numeraire invariance theorem*, a trading strategy is an arbitrage with respect to G if and only if it is an arbitrage with respect to G^Y for any deflator Y. Therefore, if there is a deflator for which G^Y is a martingale, then there is no arbitrage, since in that case

$$E_t[\theta(T)^\top V^Y(T)] = \theta(t)^\top V^Y(t)$$

for all self-financing trading strategies. Such a deflator is called a *state–price deflator.*

We assume the existence of a state–price deflator, denoted n. The dynamics of the state–price deflator can be written

$$\begin{aligned} \frac{dn(t)}{n(t)} &= \mu_n(t)\,dt + \sigma_n(t)^\top dW(t) \\ &= -r(t)\,dt - \lambda(t)^\top dW(t), \end{aligned} \tag{2.5}$$

where $r(t)$ is the interest rate and $\lambda(t)$ is the price of risk. We are free to model r and λ independently as long as there is a solution to (2.5).[2,3] Associated with any

[2]If we model r and λ as functions of a vector of Markovian state variables, X, n will not in general be a Markovian function of X. In this sense, it is less general to model n directly than to model r and λ. In particular, if one models n as a function of stationary Markovian state variable, the very-long discount bond will be the deflator asset. See Kazemi (1992).

[3]See Appendix A for a discussion of CIR's critique of the arbitrage approach to asset pricing.

state–price deflator is a *deflator–asset*: $\nu := 1/n$. By Ito's lemma, the dynamics of the deflator–asset are

$$\frac{d\nu(t)}{\nu(t)} = \left\{r(t) + \|\lambda(t)\|^2\right\} dt + \lambda(t)^\top dW(t). \tag{2.6}$$

Consider the state–price-deflated gain for a single security: $G^n = V^n + D^n$. In this case, the dynamics of the deflated gain are given by

$$\begin{aligned} dG^n(t) &= dV^n(t) + dD^n(t) \\ &= d(n(t)\,V(t)) + n(t)\,dD(t) - n(t)\,\lambda(t)^\top \sigma_D(t)\,dt \\ &= \bar{\mu}_{G^n}(t)\,dt + \bar{\sigma}_{G^n}(t)^\top dW(t), \end{aligned}$$

where

$$\begin{aligned} \bar{\mu}_{G^n}(t) &= n(t)\left\{\bar{\mu}_V(t) + \bar{\mu}_D(t) - r(t)\,V(t) - \lambda(t)^\top \left(\bar{\sigma}_V(t) + \bar{\sigma}_D(t)\right)\right\} \\ \bar{\sigma}_{G^n}(t) &= n(t)\left\{\bar{\sigma}_V(t) + \bar{\sigma}_D(t) - V(t)\,\lambda(t)\right\}. \end{aligned}$$

The absence-of-arbitrage condition states that the deflated gain is a martingale:

$$G^n(t) = E_t[G^n(\tau)], \tag{2.7}$$

for all $\tau \geq t$. We can write (2.7) as a valuation formula:

$$\begin{aligned} V(t) &= \frac{E_t[n(\tau)\,V(\tau) + D^n(\tau) - D^n(t)]}{n(t)} \\ &= E_t\left[\frac{n(\tau)}{n(t)}\,V(\tau) + \int_{s=t}^{\tau} \frac{n(s)}{n(t)} \left(\bar{\mu}_D(s) - \lambda(s)^\top \bar{\sigma}_D(s)\right) ds\right]. \end{aligned} \tag{2.8}$$

In (2.8) we see that it is the risk-adjusted value of the expected dividends that contributes to the value of the asset. The absence-of-arbitrage condition can also be expressed directly in terms of the drift of the deflated gain: $\bar{\mu}_{G^n}(t) = 0$, or

$$\left\{\bar{\mu}_V(t) - \lambda(t)^\top \bar{\sigma}_V(t)\right\} + \left\{\bar{\mu}_D(t) - \lambda(t)^\top \bar{\sigma}_D(t)\right\} = r(t)\,V(t), \tag{2.9}$$

where the risk-adjusted capital gain plus the risk-adjusted dividend equals the risk-free earnings on the value of the asset. When we introduce state variables, (2.9) becomes a partial differential equation (PDE).

If V is a positive asset and the cumulative dividend is predictable, then we have

$$\frac{dD(t)}{V(t)} = \delta(t)\,dt,$$

where δ is the *dividend rate*. In this case we can write the *deflated return* process as

$$\begin{aligned} \frac{dG^n(t)}{n(t)\,V(t)} &= \frac{dV^n(t)}{V^n(t)} + \delta(t)\,dt \\ &= \mu_{G^n}(t)\,dt + \sigma_{G^n}(t)^\top dW(t), \end{aligned}$$

where

$$\mu_{G^n}(t) = \mu_V(t) + \delta(t) - r(t) - \lambda(t)^\top \sigma_V(t)$$
$$\sigma_{G^n}(t) = \sigma_V(t) - \lambda(t),$$

and the absence-of-arbitrage condition $\mu_{G^n}(t) = 0$ becomes[4]

$$\mu_V(t) + \delta(t) = r(t) + \lambda(t)^\top \sigma_V(t). \tag{2.10}$$

An aside on the absence-of-arbitrage in a Markovian setting. Let us examine how the absence-of-arbitrage condition (2.9) or (2.10) can be put to use in a Markovian setting. We suppose there is a vector of Markovian state variables X. The dynamics of X are given by

$$dX(t) = \mu_X(X(t))\, dt + \sigma_X(X(t))^\top dW(t), \tag{2.11}$$

where $\mu_X(x)$ and $\sigma_X(x)$ are given vector and matrix functions of x. Also suppose that $r(t) = \mathbb{R}(X(t))$, $\lambda(t) = \mathbb{L}(X(t))$, $\bar{\mu}_D(t) = \mathbb{M}_D(X(t))$, and $\bar{\sigma}_D(t) = \mathbb{S}_D(X(t))$, where $\mathbb{R}(x)$ and $\mathbb{M}_D(x)$ are given scalar functions of x and $\mathbb{L}(x)$ and $\mathbb{S}_D(x)$ are given vector functions of x. Finally suppose that $V(t) = \mathbb{V}(X(t), t)$, where $\mathbb{V}(x, t)$ is an unknown scalar function x and t. Then Ito's lemma applied to $\mathbb{V}(X(t), t)$ gives $\bar{\mu}_V(t) = \mathbb{M}_V(X(t), t)$ and $\bar{\sigma}_V(t) = \mathbb{S}_V(X(t), t)$ where

$$\mathbb{M}_V(x, t) = \mu_X(x)^\top \mathbb{V}_x(x, t) + \frac{1}{2}\, \mathrm{tr}\left[\mathbb{V}_{xx}(x, t)\, \sigma_X(x)^\top \sigma_X(x)\right] - \mathbb{V}_t(x, t), \tag{2.12a}$$
$$\mathbb{S}_V(x, t) = \sigma_X(x)\, \mathbb{V}_x(x, t), \tag{2.12b}$$

and where $\mathbb{V}_t(x, t)$, $\mathbb{V}_x(x, t)$, and $\mathbb{V}_{xx}(x, t)$ are the obvious partial derivatives of $\mathbb{V}(x, t)$ and $\mathrm{tr}[a]$ is the trace of matrix a. Substituting these expressions into (2.9) produces a PDE which (along with an appropriate boundary condition—typically a terminal payoff) can be attacked by various analytical and numerical techniques.

Note that the drift of X appears only in the "risk-adjusted" drift of of V, $\mathbb{M}_V(x, t) - \mathbb{S}_V(x, t)^\top \mathbb{L}(x)$. and only in "risk-adjusted" form itself:

$$\hat{\mu}_X(x) := \mu_X(x) - \sigma_X(x)^\top \mathbb{L}(x), \tag{2.13}$$

Therefore, if the i-th state variable happens to be the value of an asset that pays no dividends, then by the absence-of-arbitrage the i-th component of $\hat{\mu}_X(x)$ is $x_i\, \mathbb{R}(x)$ and the i-th component of $\mu_X(x)$ need not be specified. Moreover, if all of the state variables are asset-values for non-dividend paying assets, then $\hat{\mu}_X(x) = x\, \mathbb{R}(x)$ and once $\hat{\mathbb{M}}_D(x) := \mathbb{M}_D(x) - \mathbb{S}_D(x)^\top \mathbb{L}(x)$ is specified, and no knowledge of $\mu_X(x)$ or $\mathbb{L}(x)$ is required.

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[4]Note that using (2.6) we can write (2.10) as

$$\{\mu_V(t) + \delta(t)\} - r(t) = \beta_{\nu,V}\, \{\mu_\nu(t) - r(t)\},$$

where

$$\beta_{\nu,V} := \frac{\lambda(t)^\top \sigma_V(t)}{\|\lambda(t)\|^2},$$

which shows that the excess total return on an asset is proportional to the excess return on the deflator–asset.

The classic example is the case where V is the value of a European call option on a stock that pays no dividends prior to the expiration of the call, where the value of the stock is the single state variable. Since the call option pays no dividends, $\bar{\mu}_D(t) \equiv \bar{\sigma}_D(t) \equiv 0$. The value of the stock is assumed to follow geometric Brownian motion, so that $\mu_X(t) = X(t)\,\mu$ and $\sigma_X(t) = X(t)\,\sigma$. The interest rate and the price of risk are assumed to be constant, $r(t) = r$ and $\lambda(t) = \lambda$. However, since the only state variable is the value of an asset that pays no dividends, neither λ nor μ need be specified: $\hat{\mu}_X(x) = x\,r$. For a call option that expires at time τ with a strike price of K, the payoff function is $\mathbb{V}(x,\tau) = \max(x - K, 0)$.

Clearly different payoff functions deliver different asset values, *i.e.* different solutions to the PDE. If one could once and for all find the value of a unit payoff in each state of the world (these are known as Arrow–Debreu state prices), then one could compute the value of any asset by adding up the values of the payoffs for that asset. Such a solution to a PDE is known as the fundamental solution or the Green's function, where payoff function is a Dirac delta function

Three fixed-income examples. It is instructive to consider three examples bond-like assets. First, consider a default-free zero-coupon bond that pays one unit at time τ. Denote its value at time t by $b(t,\tau)$. We may model the payoff to this asset as follows: It pays no dividends, $b(\tau,\tau) = 1$, $b(s,\tau) = 0$ for $s > \tau$. In this case formula (2.8) produces

$$b(t,\tau) = E_t\left[\frac{n(\tau)}{n(t)}\right]. \tag{2.14}$$

Second, consider an asset that pays a continuous risk-adjusted dividend equal to one forever. In this case (2.8) produces

$$V(t) = E_t\left[\int_{s=t}^{\infty} \frac{n(s)}{n(t)}\,ds\right] = \int_{s=t}^{\infty} E_t\left[\frac{n(s)}{n(t)}\right] ds = \int_{s=t}^{\infty} b(t,s)\,ds, \tag{2.15}$$

which shows that the value of this *perpetuity* is the integral of zero-coupon bond prices. Third, consider an asset that pays no dividends and has no instantaneous volatility, $\bar{\sigma}_V \equiv 0$. Let $\beta(t)$ denote its value at time t. Condition (2.9) implies $d\beta(t) = r(t)\,\beta(t)\,dt$, and we see that β is the value of the money-market account:

$$\beta(t) = \beta(0)\,\exp\left(\int_{s=0}^{t} r(s)\,ds\right).$$

More generally, for a positive asset that pays no dividends, we have

$$\mu_V(t) = r(t) + \lambda(t)^\top \sigma_V(t).$$

Changing the numeraire. Suppose we change the units in which asset values and cumulative dividends are measured according to (2.4). In addition, let us define a new state–price deflator as the inverse of the original deflator–asset denominated in the new units:

$$n^Y(t) := \frac{1}{Y(t)\,\nu(t)} = \frac{n(t)}{Y(t)}. \tag{2.16}$$

The dynamics of the new state–price deflator are given by

$$\frac{dn^Y(t)}{n^Y(t)} = -r_Y(t)\,dt - \lambda_Y(t)^\top dW(t),$$

where

$$r_Y(t) = r(t) + \mu_Y(t) - \lambda(t)^\top \sigma_Y(t) - \|\sigma_Y(t)\|^2 \tag{2.17a}$$

$$\lambda_Y(t) = \lambda(t) + \sigma_Y(t). \tag{2.17b}$$

Together these changes leave the expression for security prices (2.8) formally unchanged:

$$V^Y(t) = E_t\left[\frac{n^Y(\tau)}{n^Y(t)}\,V^Y(\tau) + \int_{s=t}^{\tau} \frac{n^Y(s)}{n^Y(t)}\left(\bar\mu_D^Y(s) - \lambda_Y(s)^\top \bar\sigma_D^Y(s)\right) ds\right], \tag{2.18}$$

We are free to choose the numeraire in which to model the state–price deflator. It is natural to think of Y as the exchange rate between two numeraires. Moreover, from a modeling perspective, we are free to model the state–price deflator measured in terms of two numeraires and then derive the dynamics of the deflator via the absence of arbitrage. In this case we model r, λ, r_Y, and λ_Y and solve (2.17) for μ_Y and σ_Y. This approach is useful in modeling, for example, the price level or the foreign-exchange rate as the ratio of two state–price deflators. On the other hand, if Y is the inverse of the value of a positive asset, then n^Y is just the deflated value of the asset.

Two examples are noteworthy. First, consider the value of a zero-coupon bond: $b(t,\tau) = E_t[n(\tau)/n(t)]$ in terms of the given numeraire. After deflation by Y, the value of the bond in terms of the new numeraire is the value of the deflator:

$$Y(t)\,b(t,\tau) = E_t\left[\frac{n^Y(\tau)}{n^Y(t)}\,Y(\tau)\right].$$

Second, consider the value of a perpetuity: $z(t) = \int_{s=t}^{\infty} b(t,s)\,ds$ in the given numeraire. After deflation by Y, the value of the perpetuity in terms of the new numeraire is

$$Y(t)\,z(t) = \int_{s=t}^{\infty} E_t\left[\frac{n^Y(s)}{n^Y(t)}\,Y(s)\right] ds. \tag{2.19}$$

If we define $V = Y\,z$ and $\delta = 1/z$, then (2.19) can be expressed as

$$V(t) = \int_{s=t}^{\infty} E_t\left[\frac{n^Y(s)}{n^Y(t)}\,V(s)\,\delta(s)\right] ds. \tag{2.20}$$

Equation (2.20) shows that—by changing the numeraire—the value of a perpetuity can always be expressed as the value of a positive asset with a positive, predictable dividend rate. Conversely, by choosing $Y = 1/(V\,\delta)$, the inverse of the dividend rate for such an asset can always be expressed as the value of a perpetuity.

Converting a dividend-paying asset into an asset that pays no dividends. Suppose we reinvest the dividends from an asset back into the asset. In order to do this, the asset must have a non-zero value, which we assume is positive. How will this investment's value evolve? Let θ be a scalar self-financing trading strategy:

$$\theta(t)\,V(t) = \theta(0)\,V(0) + \int_{s=t}^{t} \theta(s)\,dG(s),$$

or in differential form, $d\{\theta(t)\,V(t)\} = \theta(t)\,dG(t)$, so that

$$\frac{d\{\theta(t)\,V(t)\}}{\theta(t)\,V(t)} = \frac{dG(t)}{V(t)}. \tag{2.21}$$

Equation (2.21) implies the following dynamics for θ:

$$\frac{d\theta(t)}{\theta(t)} = \left\{\delta(t) - \sigma_D(t)^\top \sigma_V(t)\right\} dt + \sigma_D(t)^\top dW(t). \tag{2.22}$$

If the cumulative dividend is predictable, we can solve (2.22) for

$$\theta(t) = \theta(0)\,\exp\left(\int_{s=0}^{t} \delta(s)\,ds\right).$$

Note that we have an asset that has a positive value that pays no dividends. See Gilles and LeRoy (1997) for a rigorous treatment of securities of this sort.

Equivalent martingale measures. Choose a positive asset z that pays no dividends:

$$\frac{dz(t)}{z(t)} = \{r(t) + \lambda(t)^\top \sigma_z(t)\}\,dt + \sigma_z(t)^\top dW(t). \tag{2.23}$$

Define a new measure $\mathcal{Q}(z)$ via Girsanov's theorem such that

$$dW_z(t) = dW(t) + (\lambda(t) - \sigma_z(t))\,dt, \tag{2.24}$$

where W_z is a vector of standard Brownian motions under $\mathcal{Q}(z)$. The dynamics of asset prices and cumulative dividends under $\mathcal{Q}(z)$ are given by

$$\begin{aligned} dV(t) &= \left(\bar{\mu}_V(t) - (\lambda(t) - \sigma_z(t))^\top \bar{\sigma}_V(t)\right) dt + \bar{\sigma}_V(t)^\top dW_z(t) \\ &= \bar{\mu}_V^z(t)\,dt + \bar{\sigma}_V(t)^\top dW_z(t) \end{aligned}$$

and

$$\begin{aligned} dD(t) &= \left(\bar{\mu}_D(t) - (\lambda(t) - \sigma_z(t))^\top \bar{\sigma}_D(t)\right) dt + \bar{\sigma}_D(t)^\top dW_z(t) \\ &= \bar{\mu}_D^z(t)\,dt + \bar{\sigma}_D(t)^\top dW_z(t). \end{aligned}$$

In particular we have

$$\frac{dz(t)}{z(t)} = \left\{r(t) + \|\sigma_z(t)\|^2\right\} dt + \sigma_z(t)^\top dW_z(t). \tag{2.25}$$

Define $n_z := 1/z$. The dynamics of n_z under $\mathcal{Q}(z)$ are

$$\begin{aligned}\frac{dn_z(t)}{n_z(t)} &= -r(t)\,dt - \sigma_z(t)^\top dW_z(t) \\ &= -r(t)\,dt - \lambda_z(t)^\top dW_z(t).\end{aligned} \tag{2.26}$$

Note that the relative drift of n_z under $\mathcal{Q}(z)$ is the same as the relative drift of n under the physical measure.

Let $E_t^{\mathcal{Q}(z)}[\,\cdot\,]$ be the conditional expectation operator under $\mathcal{Q}(z)$, and let G^z be the deflated gain where the deflator is n_z. Then by the absence of arbitrage G^z is a martingale under $\mathcal{Q}(z)$:

$$G^z(t) = E_t^{\mathcal{Q}(z)}[G^z(\tau)]. \tag{2.27}$$

Of course, (2.27) can be written as a valuation formula:

$$V(t) = E_t^{\mathcal{Q}(z)}\left[\frac{n_z(\tau)}{n_z(t)}\,V(\tau) + \int_{s=t}^{\tau} \frac{n_z(s)}{n_z(t)}\left(\bar{\mu}_D^z(s) - \lambda_z(s)^\top \bar{\sigma}_D(s)\right) ds\right], \tag{2.28}$$

where, as it turns out,

$$\bar{\mu}_D^z(s) - \lambda_z(s)^\top \bar{\sigma}_D(s) = \bar{\mu}_D(s) - \lambda(s)^\top \bar{\sigma}_D(s).$$

We briefly discuss three equivalent martingale measures. First, let the deflator–asset be the money-market account, $z(t) = \beta(t)$. Then

$$\frac{n_z(s)}{n_z(t)} = \frac{\beta(t)}{\beta(s)} = \exp\left(-\int_{u=t}^{s} r(u)\,du\right). \tag{2.29}$$

We can write (2.28) as

$$V(t) = E_t^{\mathcal{Q}}\left[\exp\left(-\int_{s=t}^{\tau} r(u)\,du\right) V(\tau) + \int_{s=t}^{\tau} \exp\left(-\int_{u=t}^{s} r(u)\,du\right) \bar{\mu}_D^\beta(s)\,ds\right], \tag{2.30}$$

where we have abbreviated $\mathcal{Q}(\beta)$ as $\mathcal{Q}$. Note that there is no explicit reference to λ in (2.30). We can model r and $\bar{\mu}_D^\beta$ directly under this equivalent martingale measure without reference to the price of risk or the physical measure.

As an example of a financial variable that has a particularly simple representation under $\mathcal{Q}$, we examine an asset whose value is identically zero but nonetheless pays a dividend. We can interpret this asset as a continuously resettled contingent claim, such as a futures contract. Condition (2.9) says that $\bar{\mu}_D(t) = \lambda(t)^\top \bar{\sigma}_D(t)$, which implies that D is a martingale under $\mathcal{Q}$. Now suppose that

$$dD(t) = \Delta(t)\,dt + d\mathcal{F}(t),$$

and that for some fixed time $\tau \geq t$, $\mathcal{F}(\tau) = U(\tau)$. At this point, we can interpret $\mathcal{F}(t)$ as the *futures price* at time t for "delivery" of U at time τ and $\Delta(t)$ as a contractually determined dividend that accrues to the holder of the futures position.

(A standard futures contract has $\Delta \equiv 0$.) Then the martingale property of D under $\mathcal{Q}$ immediately delivers the following representation for the futures price:[5]

$$\mathcal{F}(t) = E_t^{\mathcal{Q}}\left[U(\tau) + \int_{s=t}^{\tau} \Delta(s)\, ds\right].$$

Second, let the deflator–asset be the zero-coupon bond that matures at time τ, $z(t) = b(t,\tau)$. Then

$$\frac{n_z(s)}{n_z(t)} = \frac{b(t,\tau)}{b(s,\tau)}.$$

For an asset that pays no dividends, we have the following simple pricing formula:

$$V(t) = b(t,\tau)\, E_t^{\mathcal{Q}(\tau)}\left[V(\tau)\right]. \tag{2.31}$$

since $b(\tau,\tau) = 1$ and $b(t,\tau)$ is measurable at time t. When the interest rate is deterministic, $b(t,\tau) = \exp\left(-\int_{u=t}^{\tau} r(u)\, du\right)$, and therefore $\mathcal{Q}(\tau)$ is the same as $\mathcal{Q}(\beta)$.

For example, consider $F(t,\tau)$, the forward price at time t for "delivery" of U at time τ. The payoff at time τ is $U(\tau) - F(t,\tau)$. The forward price is set at time t so that the value of the payoff is zero:

$$b(t,\tau)\, E_t^{\mathcal{Q}(\tau)}\left[U(\tau) - F(t,\tau)\right] = 0,$$

which delivers a representation for the forward price,

$$F(t,\tau) = E_t^{\mathcal{Q}(\tau)}\left[U(\tau)\right].$$

Note that when the interest rate is deterministic, the forward price equals the futures price (for a standard futures contract).

Third, assuming V is a positive asset, let $z(t) = \theta(t)\, V(t)$ where θ is the trading strategy defined in the previous section that reinvests the dividends of V back into itself. We further suppose that the cumulative dividend process is predictable. In this case we have

$$\frac{n_z(s)}{n_z(t)} = \frac{\theta(t)\, V(t)}{\theta(s)\, V(s)} = \exp\left(-\int_{u=t}^{s} \delta(u)\, du\right) \frac{V(t)}{V(s)}. \tag{2.32}$$

Using (2.32) we can write

$$\begin{aligned} &\frac{n_z(\tau)}{n_z(t)}\, V(\tau) + \int_{s=t}^{\tau} \frac{n_z(s)}{n_z(t)}\, \delta(s)\, ds \\ &\quad = V(t) \left\{ \exp\left(-\int_{s=t}^{\tau} \delta(u)\, du\right) + \int_{s=t}^{\tau} \exp\left(-\int_{u=t}^{s} \delta(u)\, du\right) \delta(s)\, ds \right\} \\ &\quad = V(t), \end{aligned} \tag{2.33}$$

and therefore (2.28) becomes the identity

$$V(t) = E_t^{\mathcal{Q}(V)}[V(t)].$$

[5] See Duffie and Stanton (1992) for a derivation of this and other results in a Markovian setting. We can easily extend our representation to allow for an unpredictable contractually determined cumulative dividend.

This identity is at the heart of the change-of-measure variance-reduction technique for Monte Carlo simulations of asset prices. It says that there is an equivalent martingale measure under which the deflated value of the asset is constant. Of course in order to change the measure we have to know $\sigma_V(t)$ which assumes we have the solution already.[6] Nevertheless, it suggests how to use related assets appropriately.

Infinitely-lived assets. For an infinitely-lived positive asset with predictable cumulative dividends, (2.8) becomes

$$V(t) = E_t\left[\int_{s=t}^{\infty} \frac{n(s)}{n(t)}\, V(s)\, \delta(s)\, ds\right]. \tag{2.34}$$

Equation (2.34) is homogeneous in V: Replacing $V(u)$ with $\alpha\, V(u)$ everywhere in (2.34) does not affect the equality. In essence, (2.34) is not an equation about V, rather it is about δ.[7] If δ is strictly positive, we can change the numeraire with $Y = 1/(V\,\delta)$, so that (2.34) can be written as

$$\frac{1}{\delta(t)} = \int_{s=t}^{\infty} E_t\left[\frac{\hat{n}(s)}{\hat{n}(t)}\right] ds = \int_{s=t}^{\infty} \hat{b}(t,s)\, ds, \tag{2.35}$$

where $\hat{n}(t) := n(t)\,\{V(t)\,\delta(t)\}$ and $\hat{b}(t,s) := E_t[\hat{n}(s)/\hat{n}(t)]$. Equation (2.35) shows that under these circumstances the inverse of the dividend rate can be expressed as the value of a perpetuity—where the state–price deflator is the deflated value of the dividend.[8]

Since the dividend rate is positive, we can write its dynamics as

$$\frac{d\delta(t)}{\delta(t)} = \mu_\delta(t)\, dt + \sigma_\delta(t)^\top dW(t).$$

Now let $z(t) := 1/\delta(t)$, so that $dz(t)/z(t) = \mu_z(t)\, dt + \sigma_z(t)^\top dW(t)$, where

$$\begin{aligned} \mu_z(t) &= -\mu_\delta(t) + \|\sigma_\delta(t)\|^2 \\ \sigma_z(t) &= -\sigma_\delta(t). \end{aligned}$$

Since $z(t)$ is the value of a perpetuity, the perpetuity's dividend rate is $1/z(t)$. In terms of the new numeraire, the absence-of-arbitrage condition (2.10) is

$$\mu_z(t) + \frac{1}{z(t)} = \hat{r}(t) + \hat{\lambda}(t)^\top \sigma_z(t), \tag{2.36}$$

where

$$\frac{d\hat{n}(t)}{\hat{n}(t)} = -\hat{r}(t)\, dt - \hat{\lambda}(t)^\top dW(t).$$

If we choose to introduce the dynamics of the state variables into the model via $\hat{r}$ and $\hat{\lambda}$, then (2.36) becomes the partial differential equation that must be solved to

[6]Note that the relative volatility of $\theta\, V$ is the same as the relative volatility of V when the cumulative dividend is predictable.

[7]This is related to the fact that one cannot tell whether we have an endowment economy or a production economy.

[8]Note however that $d\hat{n}(t) - dD^n(t) = \{V(t)\,\delta(t)\}\, dn(t) \neq 0$.

find z (and hence δ) as a function of the state variables. The route to solving (2.36) is to solve the PDE for bond prices, $\hat{b}(t,\tau)$ and integrate:

$$\hat{\mu}_b(t,\tau) - \hat{\lambda}(t)^\top \hat{\sigma}_b(t,\tau) = \hat{r}(t),$$

where

$$\frac{d\hat{b}(t,\tau)}{\hat{b}(t,\tau)} = \hat{\mu}_b(t,\tau)\,dt + \hat{\sigma}_b(t,\tau)^\top dW(t).$$

At this point, we change our perspective a bit, and interpret the deflated value of the asset, $V^n(t) = n(t)\,V(t)$, as a state–price deflator for payoffs denominated in units of the asset's value. In other words, let $Y = 1/V$. The dynamics of this state–price deflator can be written

$$\frac{dV^n(t)}{V^n(t)} = -r_V(t)\,dt - \lambda_V(t)^\top dW(t), \tag{2.37a}$$

where

$$r_V(t) = \delta(t) \tag{2.37b}$$

$$\lambda_V(t) = \lambda(t) - \sigma_V(t). \tag{2.37c}$$

If we choose introduce the dynamics of the state variables into the model via r_V and λ_V, then there is no differential equation to solve, and the absence-of-arbitrage delivers the dynamics V directly via $V = V^n/n$.

3. Utility and general equilibrium

Duffie and Skiadas (1994) demonstrate a very useful equivalence in a in a very general setting. They consider

> a continuous-time security market where prices are modeled by semi-martingales (allowing for jumps, and therefore incorporating discrete time as a special case). The underlying information filtration is general, and the market is not necessarily complete.

In such a setting, they show that

> the first-order conditions for optimality of an agent maximizing a 'smooth' utility can be formulated as the martingale property of prices, after normalization by a 'state–price' process.

In addition, they show that the state–price deflator

> is given explicitly in terms of the agent's utility gradient, which is in turn computed in closed-form for a wide class of dynamics utilities, including the stochastic differential utility . . . .

Stochastic Differential Utility. Here we introduce the preferences of the representative agent. As explained by Duffie and Epstein (1992a) and Duffie and Epstein (1992b), stochastic differential utility SDU can be represented by a pair of functions (f, A) called an *aggregator*. The functions f and A can be chosen to capture separately attitudes toward intertemporal substitution and attitudes toward risk. Hypothetical experiments can be conducted, for example, by fixing f and varying

A to study the effect of increasing risk aversion. Importantly, though, there exists a normalized form of SDU $(\bar{f}, \bar{A})$ where $\bar{A} \equiv 0$. By using the normalized aggregator significant analytical simplification is achieved, although the convenient separation referred to above is lost since both aspects of preferences are combined in $\bar{f}$. Since we only deal with the normalized aggregator, we drop the bar for notational simplicity.

When the normalized aggregator is used, the recursive nature of utility is expressed in the following representation:

$$\mathcal{V}(t) = E_t \left[\int_{s=t}^{\infty} f(c(s), \mathcal{V}(s))\, ds \right], \tag{3.1}$$

where $\mathcal{V}(t)$ is the utility at time t of the consumption process c. The dynamics of $\mathcal{V}$ are given by

$$d\mathcal{V}(t) = \mu_{\mathcal{V}}(t)\, dt + \sigma_{\mathcal{V}}(t)^{\top} dW(t),$$

where

$$\mu_{\mathcal{V}}(t) = -f(c(t), \mathcal{V}(t)). \tag{3.2}$$

Duffie and Skiadas (1994) show that the state–price deflator is given by

$$n(t) = \exp\left\{ \int_{s=0}^{t} f_v(c(s), \mathcal{V}(s))\, ds \right\} f_c(c(t), \mathcal{V}(t)). \tag{3.3}$$

where f_c and f_v are the the partial derivatives of f and where c is the *optimal* rate of consumption. Suppressing arguments, let the dynamics of f_c be given by

$$\frac{df_c}{f_c} = \mu_{f_c}\, dt + \sigma_{f_c}^{\top} dW. \tag{3.4}$$

Then, given (3.3) and (3.4), we can write

$$\frac{dn}{n} = \frac{df_c}{f_c} + f_v\, dt,$$

so that

$$r = -(\mu_{f_c} + f_v) \tag{3.5a}$$

$$\lambda = -\sigma_{f_c}. \tag{3.5b}$$

Equations (3.5) are general equilibrium expressions for the interest rate and the price of risk in terms of the utility function and the dynamics of optimal consumption. Inserting (3.5) into (2.9) produces a general-equilibrium valuation equation. However, in order to follow this route, we need to be able to evaluate (3.3) given the process for consumption. Equation (3.2) is the key.

Equation (3.2) is the central restriction for a model of stochastic differential utility. In a Markovian setting, (3.2) becomes a PDE. Duffie and Lions (1992) address the existence and uniqueness of $\mathcal{V}$ when c is modeled in terms of state variables (where c itself may be a state variable). For example, let $\mathcal{V}(t) = H(X(t), c(t))$ where X is a vector of Markovian state variables that describe the dynamics of c and $H(x, c)$ is an unknown function. The joint dynamics of c and X define a differential operator $\mathcal{D}$ such that $\mu_V(t) = \mathcal{D}H(X(t), c(t))$. With this setup, (3.2) becomes the PDE

$$\mathcal{D}H(x, c) = -f(c, H(x, c)). \tag{3.6}$$

One can insert the solution to (3.6) for H into (3.3) and use Ito's lemma to deliver expressions for r and λ in terms of the state variables. These expressions for the basis for the valuation equation—*i.e.*, the absence-of-arbitrage equation, for example (2.9). When this valuation equation is applied to the capital stock (wealth), it delivers the optimal relationship between capital and consumption.

A different approach is to transform (3.2) directly into the Bellman equation for SDU by imposing conditions for the optimality of consumption relative to wealth (by imposing the envelope condition that the marginal utility of consumption equal the marginal utility of wealth). We pursue this approach below.

Special case: Time-separable utility. Time-separable utility is a special case of SDU where one does not need to solve (3.6) in order to find the state–price deflator. With time-separable utility, utility can be written as

$$\mathcal{V}(t) = E_t\left[\int_{s=t}^{\infty} e^{-\theta\,(s-t)}\,u(c(t))\,ds\right] = \int_{s=t}^{\infty} e^{-\theta\,(s-t)}\,E_t\left[u(c(t))\right]\,ds. \tag{3.7}$$

where θ is the rate of time preference and $c(t)$ is the rate of consumption at time t. Applying Ito's lemma to (3.7) (and using the fact that conditional expectations are martingales) produces

$$d\mathcal{V}(t) = \big(\theta\,\mathcal{V}(t) - u(c(t))\big)\,dt + \sigma_{\mathcal{V}}(t)^\top dW(t). \tag{3.8}$$

Since $\mu_{\mathcal{V}}(t) = -f(c(t), \mathcal{V}(t))$, (3.8) shows that with time-separable preferences the normalized aggregator can be written

$$f(c, v) = u(c) - \theta\,v,$$

so that $f_v(c, v) = -\theta$ and $f_c(c, v) = u'(c)$.

In this case, the state–price deflator is given by

$$n(t) = e^{-\theta\,t}\,u'(c(t)), \tag{3.9}$$

where $c(t)$ is the *optimal* rate of consumption. Therefore, in terms of the dynamics of the growth rate of optimal consumption,

$$d\log(c(t)) = \widetilde{\mu}_c(t)\,dt + \sigma_c(t)^\top dW(t),$$

applying Ito's lemma to (3.9) we get

$$r(t) = \theta - \frac{c(t)\,u''(c(t))}{u'(c(t))}\left(\widetilde{\mu}_c(t) + \frac{1}{2}\,\|\sigma_c(t)\|^2\right) - \frac{c(t)^2\,u'''(c(t))}{u'(c(t))}\,\frac{1}{2}\,\|\sigma_c(t)\|^2 \tag{3.10a}$$

$$\lambda(t) = -\frac{c(t)\,u''(c(t))}{u'(c(t))}\,\sigma_c(t). \tag{3.10b}$$

Inserting (3.10) into (2.9) delivers a general-equilibrium valuation formula. This is in essence what Bakshi and Chen (1997) do. Note that if we change both $u(\,\cdot\,)$ and $d\log(c(t))$ in such a way as to leave $dn(t)/n(t)$ unchanged, then the two economies will have identical interest-rate and price-of-risk processes and hence identical valuation formulas.[9]

The capital stock, consumption, and the capital account. Our central assumption here is that the dividend that accrues to the owners of the the aggregate capital stock is aggregate consumption. Let $c(t)$ be the aggregate rate of consumption and $k(t)$ be the value of the aggregate capital stock. We treat the aggregate capital stock as synonymous with wealth. Define $\omega(t) := c(t)/k(t)$ as the consumption–capital ratio. In this case we can write the value of the capital stock as follows:

$$k(t) = E_t\left[\int_{s=t}^{\infty}\left(\frac{n(s)}{n(t)}\right) k(s)\,\omega(s)\,ds\right]. \tag{3.11}$$

Given the framework outlined in Section 2 in and given

$$\frac{dk(t)}{k(t)} = \mu_k(t)\,dt + \sigma_k(t)^\top dW(t), \tag{3.12}$$

three useful representations follow immediately. First, we have the absence-of-arbitrage condition for the capital stock:

$$\mu_k(t) + \omega(t) = r(t) + \lambda(t)^\top \sigma_k(t). \tag{3.13}$$

Second, we can let the deflated value of the capital stock be a state–price deflator: $m_k(t) := m(t)\,k(t)$. Given (2.37), we have

$$\frac{dm_k(t)}{m_k(t)} = -\omega(t)\,dt + (\sigma_k(t) - \lambda(t))^\top dW(t). \tag{3.14}$$

We see that we can model ω and $\lambda - \sigma_k$ independently. Third, we can write the inverse of the consumption–capital ratio, as a perpetuity:

$$\frac{1}{\omega(t)} = E_t\left[\int_{s=t}^{\infty}\frac{\hat{m}(s)}{\hat{m}(t)}\,ds\right] = \int_{s=t}^{\infty}\hat{b}(t,s)\,ds,$$

where $\hat{m}(t) := m(t)\,k(t)\,\omega(t) = m(t)\,c(t)$ and $\hat{b}(t,T) = E_t[\hat{m}(T)/\hat{m}(t)]$. Let $\pi(t) := 1/\omega(t) = k(t)/c(t)$ be the capital–consumption ratio. Then the absence-of-arbitrage condition for the capital stock can be written

$$\mu_\pi(t) + \frac{1}{\pi(t)} = \hat{r}(t) + \hat{\lambda}(t)^\top \sigma_\pi(t). \tag{3.15}$$

We define the *capital account* as follows:

$$\phi(t) = k(t)\,\exp\left(\int_{s=0}^{t}\omega(s)\,ds\right). \tag{3.16}$$

Therefore we have

$$\frac{d\phi(t)}{\phi(t)} = \frac{dk(t)}{k(t)} + \omega(t)\,dt \tag{3.17}$$

[9]This point is illustrated (with a slight error) by Bakshi and Chen (1997).

and

$$\mu_\phi(t) = r(t) + \lambda(t)^\top \sigma_\phi(t) \tag{3.18a}$$

$$\sigma_\phi(t) = \sigma_k(t). \tag{3.18b}$$

We can interpret $d\phi(t)/\phi(t)$ as the return on optimally invested wealth in either an endowment economy or a production economy with linear technology. In a production economy, we require linear technology to guarantee that the dividend rate only involves consumption and not the difference between average and marginal productivity. In an endowment economy, even though the "optimal portfolio" is given exogenously, the interpretation is still valid.

Kreps–Porteus SDU. For Kreps–Porteus SDU, the normalized form of the aggregator is[10]

$$f(c, v) = \frac{\theta\, \alpha \left(c^{1-1/\eta}\, \alpha^{-1/\delta} - 1\right)}{1 - 1/\eta}, \quad \alpha = 1 + (1-\gamma)\, v, \tag{3.19}$$

where θ, η, and γ are constant parameters, and the parameter δ is defined by

$$\delta := \frac{1-\gamma}{1 - 1/\eta}.$$

For the cases $\gamma = 1$ and $\eta = 1$, we get the normalized aggregator by taking a limit in (3.19). As shown by Duffie and Epstein (1992a), these preferences allow a disentangling of attitudes toward risk from attitudes toward intertemporal substitution. In this parameterization, $\eta > 0$ is the elasticity of intertemporal substitution, $\gamma > 0$ is the coefficient of relative risk aversion and, $\theta > 0$ is the rate of time preference. When $\gamma\, \eta = 1$, in which case $\delta = 1$, Kreps–Porteus SDU specializes to standard time-separable preferences with power utility, as we will see below.

On the optimal path, consumption and continuation utility can be written as follows (where $\mathcal{V}(t)$ is replaced by the obvious limit when $\gamma = 1$):

$$c(t) = \theta^\eta\, \psi(t)^{1-\eta}\, k(t) \tag{3.20a}$$

$$\mathcal{V}(t) = \frac{(\psi(t)\, k(t))^{1-\gamma} - 1}{1-\gamma}, \tag{3.20b}$$

The process ψ is related to the marginal utility of wealth.[11] The optimality of (3.20) requires that (3.2) hold identically for some processes k and ψ. Along the optimal path, the normalized aggregator is given by

$$f(c(t), \mathcal{V}(t)) = (\psi(t)\, k(t))^{1-\gamma} \left\{ \frac{\theta^\eta\, \psi(t)^{1-\eta} - \theta}{1 - 1/\eta} \right\}. \tag{3.21}$$

We find $\mu_{\mathcal{V}}$ as follows. The consumption–capital ratio can be expressed in terms of ψ:

$$\omega(t) = \theta^\eta\, \psi(t)^{1-\eta}, \tag{3.22}$$

[10]The functional form of the aggregator comes from the unnumbered equation in the middle of p. 420 in Duffie and Epstein (1992a). Their ρ is $1 - 1/\eta$, their α is $1-\gamma$, and their β is θ. Note that we have replaced $(1-\gamma)\, v$ by $1 + (1-\gamma)\, v$ in order to take the limit as $\gamma \to 1$.

[11]Letting $\hat{\mathcal{V}}(t) := ((1-\gamma)\, \mathcal{V}(t) + 1)^{1/(1-\gamma)}$, $\partial \hat{\mathcal{V}}(t)/\partial\, k(t) = \psi(t)$.

which can be used to write

$$d\log(k(t)) = d\log(\phi(t)) - \theta^\eta\,\psi((t)^{1-\eta}\,dt,$$

where

$$d\log(\psi(t)) = \widetilde{\mu}_\psi(t)\,dt + \sigma_\psi(t)^\top dW(t).$$

Now applying Ito's lemma to (3.20b) produces

$$\mu_{\mathcal{V}}(t) = (\psi(t)\,k(t))^{1-\gamma}\left\{\widetilde{\mu}_\phi(t) + \widetilde{\mu}_\psi(t) + (1-\gamma)\,\frac{1}{2}\,\|\sigma_\phi(t)+\sigma_\psi(t)\|^2 - \theta^\eta\,\psi(t)^{1-\eta}\right\}. \tag{3.23}$$

Finally, substituting (3.21) and (3.23) into (3.2) produces

$$\frac{\theta^\eta\,\psi(t)^{1-\eta} - \eta\,\theta}{\eta-1} + \widetilde{\mu}_\phi(t) + \widetilde{\mu}_\psi(t) + (1-\gamma)\,\frac{1}{2}\,\|\sigma_\phi(t)+\sigma_\psi(t)\|^2 = 0. \tag{3.24}$$

Equation (3.24) is the central restriction for Kreps–Porteus SDU. We see that k does not enter (3.24). Thus, for any ψ that satisfies (3.24), (3.20) is optimal. Moreover, given the the equivalence established by Duffie and Skiadas (1994) between (i) the first order conditions for the optimality of consumption and (ii) the martingale property of deflated asset prices, (3.24) is equivalent to the absence-of-arbitrage condition for the capital account. This can be verified directly once the state–price deflator is derived (see (3.27) below) by showing that $m(t)\,\phi(t)$ is a martingale given (3.20) if and only if (3.24) holds.

In a Markovian setting, (3.24) is the Bellman equation—the PDE that ψ must satisfy as a function of the state variables. We now show that in a Markovian setting the first-order condition for the Bellman equation (the envelope condition) is satisfied. Let X be a vector of state variables. The optimal rate of consumption and the maximized value of utility depend on the state variables and wealth as follows: $c(t) = C(X(t), k(t))$ and $\mathcal{V}(t) = J(X(t), k(t))$, where

$$C(x,k) = \theta^\eta\,\Psi(x)^{1-\eta}\,k \quad \text{and} \quad J(x,k) = \frac{(\Psi(x)\,k)^{1-\gamma}-1}{1-\gamma}. \tag{3.25}$$

Let J_k and f_c denote the obvious partial derivatives. One can verify that (3.19) and (3.25) satisfy the envelope condition *identically*:

$$J_k(x,k) \equiv f_c(C(x,k), J(x,k)) = \Psi(x)^{1-\gamma}\,k^\gamma.$$

Therefore, the solution to the optimum control problem is reduced to finding the function $\Psi(x)$ that satisfies a Markovian version of (3.24) (in which $\widetilde{\mu}_\phi(t)$ and $\sigma_\phi(t)$ are determined by functions of the same state variables).

The state–price deflator for Kreps–Porteus SDU. Given (3.20), the partial derivatives of f on the optimal path can be written as

$$f_c(c(t),\mathcal{V}(t)) = \psi(t)^{1-\gamma}\,k(t)^{-\gamma} \tag{3.26a}$$

$$f_v(c(t),\mathcal{V}(t)) = \frac{(1-\gamma)\,\theta\,\eta + (\gamma\,\eta-1)\,\theta^\eta\,\psi(t)^{1-\eta}}{1-\eta}. \tag{3.26b}$$

We obtain a representation of the state-price deflator simply by inserting (3.26) into (3.3):

$$m(t) = \exp\left(\int_{s=0}^{t} \frac{(1-\gamma)\,\theta\,\eta + (\gamma\,\eta - 1)\,\theta^{\eta}\,\psi(s)^{1-\eta}}{1-\eta}\,ds\right) \psi(t)^{1-\gamma}\,k(t)^{-\gamma}. \tag{3.27}$$

We can solve for the interest rate and the price of risk by applying Ito's lemma to (3.27) and using (2.5):

$$r(t) = \gamma\,\widetilde{\mu}_{\phi}(t) + (\gamma-1)\,\widetilde{\mu}_{\psi}(t) + \frac{(1-\gamma)\,(\theta^{\eta}\,\psi(t)^{1-\eta} - \eta\,\theta)}{1-\eta} - \frac{1}{2}\,\|\lambda(t)\|^2 \tag{3.28}$$

$$\lambda(t) = \gamma\,\sigma_{\phi}(t) + (\gamma-1)\,\sigma_{\psi}(t). \tag{3.29}$$

Equations (3.28) impound in asset prices all the characteristics of preferences we need to solve for the equilibrium. In other words, once we have (3.28), the we do not need the value function itself for asset pricing or for the dynamics of consumption.

The question remains as to how to find ψ. We pursue two approaches.

Using (3.22) and (3.27), the dynamics of the deflated value of the capital stock (3.14) can be written

$$\frac{dm_k(t)}{m_k(t)} = -\theta^{\eta}\,\psi(t)^{1-\eta}\,dt + (1-\gamma)\,(\sigma_{\psi}(t) + \sigma_{\phi}(t))^{\top}\,dW(t),$$

Since $m_k(t) = m(t)\,k(t)$ can be interpreted as a state–price deflator, we are free to model its drift and diffusion independently. Therefore, we are free to model ψ and σ_{ϕ}. Having done so, (3.24) can be solved *algebraically* for $\widetilde{\mu}_{\phi}$. Moreover, we can solve for the dynamics of consumption growth and the state–price deflator:

$$\widetilde{\mu}_c(t) = \frac{\eta\,\theta - \eta\,\theta^{\eta}\,\psi(t)^{1-\eta}}{\eta-1} - \eta\,\widetilde{\mu}_{\psi}(t) + \quad (\gamma-1)\,\frac{1}{2}\,\|\sigma_{\phi}(t) + \sigma_{\psi}(t)\|^2 \tag{3.30a}$$

$$\widetilde{\mu}_{\phi}(t) = \frac{\eta\,\theta - \quad \theta^{\eta}\,\psi(t)^{1-\eta}}{\eta-1} - \quad \widetilde{\mu}_{\psi}(t) + \quad (\gamma-1)\,\frac{1}{2}\,\|\sigma_{\phi}(t) + \sigma_{\psi}(t)\|^2 \tag{3.30b}$$

$$r(t) = \frac{\eta\,\theta - \quad \theta^{\eta}\,\psi(t)^{1-\eta}}{\eta-1} - \quad \widetilde{\mu}_{\psi}(t) + \gamma\,(\gamma-1)\,\frac{1}{2}\,\|\sigma_{\phi}(t) + \sigma_{\psi}(t)\|^2 - \frac{1}{2}\,\|\lambda(t)\|^2 \tag{3.30c}$$

$$\sigma_c(t) = \quad \sigma_{\phi}(t) + (1-\eta)\,\sigma_{\psi}(t) \tag{3.30d}$$

$$\sigma_{\phi}(t) = \quad \sigma_{\phi}(t) \tag{3.30e}$$

$$\lambda(t) = \gamma\,\sigma_{\phi}(t) + (\gamma-1)\,\sigma_{\psi}(t). \tag{3.30f}$$

Note that

$$\lim_{\eta\to 1} \frac{\eta\,\theta - \eta\,\theta^{\eta}\,\psi(t)^{1-\eta}}{\eta-1} = \theta\,\log\left(\frac{\psi(t)}{\theta}\right) \tag{3.31a}$$

$$\lim_{\eta\to 1} \frac{\eta\,\theta - \quad \theta^{\eta}\,\psi(t)^{1-\eta}}{\eta-1} = \theta\,\log\left(\frac{\psi(t)}{\theta}\right) + \theta. \tag{3.31b}$$

Equations (3.30) give a complete algebraic solution to the model if we choose to introduce the state variables directly though ψ and σ_{ϕ}. We are free to choose how ψ and σ_{ϕ} behave and free to choose the three preference parameters θ, γ, and η.

In particular, let $\psi(t) = \Psi(X(t))$ and $\sigma_\phi(t) = \sigma_\Phi(X(t))$, where $\Psi(x)$ and $\sigma_\Phi(x)$ are arbitrarily chosen functions. Specify the dynamics of the Markovian state variables, X. Ito's lemma then delivers $\widetilde{\mu}_\psi(t)$ and $\sigma_\psi(t)$ as functions of the state variables (and their dynamics). Equation (3.30) determines the rest of the model algebraically.[12] Note that given the solution just described and recalling (3.25), we now can price any contingent claim along the lines of Cox, Ingersoll, Jr., and Ross (1985a), applying the fundamental valuation equation of their Theorem 3.

The perpetuity equation. If we choose instead to introduce the state variables via the dynamics of the capital account or consumption or the state–price deflator, then we will have to solve (3.24) as a PDE. Recall the absence-of-arbitrage condition for the capital-stock-as-perpetuity (3.15), which we repeat here for convenience:

$$\widetilde{\mu}_\pi(t) + \frac{1}{\pi(t)} = \hat{r}(t) + \hat{\lambda}(t)^\top \sigma_\pi(t). \tag{3.15}$$

When $\eta = 1$, (3.15) devolves to an identity. Otherwise, it forms the basis for finding tractable solutions. Let y be the forcing process, where y is either ϕ or c or $1/m$. Then we can write

$$\hat{r}(t) = \left\{ d_0 + d_1\, \widetilde{\mu}_y(t) + d_1\, d_2\, \frac{1}{2}\, \|\sigma_y(t)\|^2 \right\} + \left\{ \varepsilon\, d_1\, \frac{1}{2} \left\| \frac{\sigma_\pi(t)}{d_1} \right\|^2 \right\} \tag{3.32a}$$

$$\hat{\lambda}(t) = -d_2\, \sigma_y(t). \tag{3.32b}$$

where d_1 depends only on η and d_2 depends only on γ. (See Table 1 for the particular values.) Given $\widetilde{\mu}_y$ and σ_y, we can solve (3.15) for π. The remaining drifts and diffusions can be found via (3.30).[13]

$y(t)$	$\widetilde{\mu}_y(t)$	$\sigma_y(t)$	d_0	d_1	d_2	d_3	$\varepsilon = d_1 + d_2$
$\phi(t)$	$\widetilde{\mu}_\phi(t)$	$\sigma_\phi(t)$	$\eta\,\theta$	$1-\eta$	$1-\gamma$	-1	$2-\eta-\gamma$
$c(t)$	$\widetilde{\mu}_c(t)$	$\sigma_c(t)$	θ	$\frac{1-\eta}{\eta}$	$1-\gamma$	0	$\frac{1}{\eta}-\gamma$
$1/m(t)$	$r(t) + \frac{1}{2}\,\|\lambda(t)\|^2$	$\lambda(t)$	$\eta\,\theta$	$1-\eta$	$\frac{1-\gamma}{\gamma}$	-1	$\frac{1}{\gamma}-\eta$

TABLE 1. The coefficients of Equations (3.32) and (3.35) in terms of the preference parameters.

When $\eta = 1$, we take a different tack. Note that

$$\lim_{\eta \to 1} \frac{\theta^\eta\, \psi(t)^{1-\eta} - \eta\,\theta}{\eta - 1} = -\theta \left\{ 1 + \log\left(\frac{\psi(t)}{\theta} \right) \right\}. \tag{3.33}$$

[12]Note that although we can get both r and λ algebraically from (3.30), will still need to use them to solve the bond-price PDE to get the term structure. Therefore, for some purposes, it may be preferable to model r and λ directly in terms of the state variables.

[13]It turns out that other drifts and diffusions can always be expressed in terms of $\widetilde{\mu}_y$, σ_y, and σ_π without $\widetilde{\mu}_\pi$.

Therefore we can write (3.24) as

$$-\theta\,(1+\log(\psi(t)/\theta)) + \widetilde{\mu}_\psi(t) + (1-\gamma)\,\frac{1}{2}\,\|\sigma_\phi(t)+\sigma_\psi(t)\|^2 = 0. \tag{3.34}$$

In terms of y, we can write (3.34) as

$$\theta\,\log(\psi(t)/\theta) = d_3\,\theta + \widetilde{\mu}_y(t) + \widetilde{\mu}_\psi(t) + \frac{d_2}{2}\,\|\sigma_y(t)+\sigma_\psi(t)\|^2, \tag{3.35}$$

where d_2 and d_3 are given in Table 1.

The value of currency and the nominal interest rate. If we put real balances in a time-separable utility function, we have

$$f(c,m,v) = u(c,m) - \theta\,v,$$

so that $f_v(c,v) = -\theta$, $f_c(c,v) = u_c(c,m)$, and the state–price deflator is given by

$$n(t) = e^{-\theta\,t}\,u_c(c(t),m(t)).$$

In addition, we have the first-order condition that the rental rate of money equal the marginal rate of substitution:

$$R(t) = \frac{f_m(c,v)}{f_c(c,v)} = \frac{u_m(c,m)}{u_c(c,m)},$$

where R is the instantaneous nominal interest rate.[14]

Now consider the value of currency, $V(t) = 1/P(t)$. In this case we model the nominal state–price deflator:

$$n_1(t) := \frac{n(t)}{P(t)},$$

where

$$\frac{dn_1(t)}{n_1(t)} = -R(t)\,dt - \Lambda(t)^\top dW(t).$$

The dividend rate is the nominal interest rate $R(t)$ and the nominal price of risk is the sum of the real price of risk and inflation volatility, $\Lambda(t) = \lambda(t) + \sigma_P(t)$. The dynamics of inflation are given by applying Ito's lemma to $P(t) = n(t)/n_1(t)$:

$$d\log(P(t)) = \pi(t)\,dt + \sigma_P(t)^\top dW(t),$$

where

$$\begin{aligned}
\pi(t) &= R(t) - r(t) + \frac{1}{2}\left(\|\Lambda(t)\|^2 - \|\lambda(t)\|^2\right)\\
\sigma_P(t) &= \Lambda(t) - \lambda(t).
\end{aligned}$$

[14] See Duffie and Epstein (1992a) for a discussion of the multiple-goods case.

With money in a time–separable utility function, we have[15,16]

$$R(t) = \frac{u_m\left(c(t), \frac{M(t)}{P(t)}\right)}{u_c\left(c(t), \frac{M(t)}{P(t)}\right)}. \tag{3.36}$$

Given that we have already solved for the dynamics of $R(t)$, $c(t)$, and $P(t)$, we can apply Ito's lemma to this condition to recover the dynamics of equilibrium money money supply. For example let

$$u(c, m) = \gamma \left(\frac{c^{1-\eta} - 1}{1-\eta}\right) + (1-\gamma)\left(\frac{m^{1-\psi} - 1}{1-\psi}\right), \tag{3.37}$$

so that

$$u_c(c, m) = \gamma\, c^{-\eta} \tag{3.38a}$$

$$u_m(c, m) = (1-\gamma)\, m^{-\psi}. \tag{3.38b}$$

In this case we have $n(t) = e^{-\theta\, t}\, c(t)^{-\eta}$, which is a special case of the state–price deflator presented above with $\delta = 1$.

With this utility function, (3.36) becomes

$$\frac{M(t)}{P(t)} = \left(\frac{1-\gamma}{\gamma}\right)^{1/\psi} \frac{c(t)^{\eta/\psi}}{R(t)^{1/\psi}}, \tag{3.39}$$

and we see that the elasticity of demand for real balances with respect to consumption and the nominal interest rate are given by η/ψ and $-1/\psi$ respectively. Applying Ito's lemma to (3.36) in this case produces

$$\widetilde{\mu}_M(t) = \frac{1}{\psi}\left(R(t) - \theta - \widetilde{\mu}_R(t) + \frac{1}{2}\,\|\Lambda(t)\|^2\right) + \left(\frac{\psi - 1}{\psi}\right)\pi(t) \tag{3.40a}$$

$$\sigma_M(t) = \frac{1}{\psi}\left(\Lambda(t) - \sigma_R(t)\right) + \left(\frac{\psi-1}{\psi}\right)\sigma_P(t), \tag{3.40b}$$

where

$$d\log(R(t)) = \widetilde{\mu}_R(t)\, dt + \sigma_R(t)^\top dW(t).$$

Appendix A. A note on CIR's critique of the arbitrage approach to bond pricing

Cox, Ingersoll, Jr., and Ross (1985b) critique what they call "bond pricing by arbitrage methods." They focus their discussion on the PDE for bond prices where there is a single state variable, the instantaneous interest rate r. The dynamics of r under the physical measure given by

$$dr = \kappa\,(\theta - r)\, dt + \sigma\,\sqrt{r}\, dW(t).$$

[15]See Bakshi and Chen (1996) for an analysis of this model.
[16]See Appendix B for an example of money in a recursive utility function.

The PDE is given as their equation (33),

$$\frac{1}{2}\,\sigma^2\,r\,P_{rr} + \kappa\,(\theta - r)\,P_r + P_t - r\,P = Y(r,t,T).$$

where P is the price of a zero-coupon bond of that matures at time T and $Y(r,t,T)$ is "the excess expected return on a bond with maturity date T." Clearly, Y is the product of the (absolute) volatility of the bond price, $\sigma\sqrt{r}\,P_r$, times the price of risk, $\lambda(r)$.[17] CIR write

$$Y(r,t,T) = \psi(r)\,P_r,$$

which implies

$$\psi(r) = \sigma\sqrt{r}\,\lambda(r).$$

In their example, CIR set $\psi(r) = \psi_0 + \psi_1\,r$, where ψ_0 and ψ_1 are non-zero constants. We can see the problem by solving for $\lambda(r)$:

$$\lambda(r) = \frac{\psi(r)}{\sigma\sqrt{r}} = \frac{\psi_0 + \psi_1\,r}{\sigma\sqrt{r}}.$$

Note that

$$\lim_{r\to 0}\,|\lambda(r)| = \infty.$$

Clearly this is a bad modeling choice. Indeed, one is not free to model the price of risk in such a way as to preclude the existence of the state–price deflator. Note that CIR *do not* show that one cannot model r and λ independently. In fact, one can.

Appendix B. Money in a recursive utility function

Refer to Fisher and Gilles (1997). Let

$$\bar f(c,m,v) = \frac{\theta\left(g(m)\,c^{1-1/\eta} - \{1 + (1-\gamma)\,v\}^{1/\delta}\right)}{(1-1/\eta)\,\{1+(1-\gamma)\,v\}^{1/\delta-1}},$$

and keep

$$c(t) = \omega(t)\,k(t) \tag{B.1a}$$

$$V(t) = \frac{(\zeta(t)\,k(t))^{1-\gamma} - 1}{1-\gamma}. \tag{B.1b}$$

But let the consumption–capital ratio $\omega(t)$ is defined in terms of the function $\zeta(t)$ by:

$$\omega(t) := g(m)^\eta\,\theta^\eta\,\zeta(t)^{1-\eta}. \tag{B.2}$$

Given (B.1), the partial derivatives of $\bar f$ on the optimal path can be written as

$$\bar f_c(c(t),V(t)) = g(m)\,\zeta(t)^{1-\gamma}\,k(t)^{-\gamma} \tag{B.3a}$$

and

$$\bar f_v(c(t),V(t)) = -\delta\,\theta + (\delta - 1)\,\omega(t). \tag{B.3b}$$

[17] This is *not* CIR's parameter λ.

The state–price deflator is given by

$$n(t) = e^{-\theta\,\delta\,t}\,c(t)^{-\delta/\eta}\,\phi(t)^{\delta-1}\,g(m(t)).$$

The first-order condition for money, $R = \bar{f}_m/\bar{f}_c$, is

$$R(t) = \frac{c(t)\,g'(m(t))}{(1-1/\eta)\,g(m(t))}.$$

A natural choice for g is $g(m) = m^{1-\psi}$, in which case we have

$$n(t) = e^{-\theta\,\delta\,t}\,c(t)^{-\delta/\eta}\,\phi(t)^{\delta-1}\,m(t)^{1-\psi}$$
$$R(t) = \left(\frac{1-\psi}{1-1/\eta}\right)\frac{c(t)}{m(t)}.$$

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